

# Hidden Translation and Orbit Coset in Quantum Computing\*

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## Abstract

We give efficient quantum algorithms for HIDDEN TRANSLATION and HIDDEN SUBGROUP in a large class of non-abelian groups including solvable groups of bounded exponent and of bounded derived series. Our algorithms are recursive. For the base case, we solve efficiently HIDDEN TRANSLATION in  $\mathbb{Z}_p^n$ , whenever  $p$  is a fixed prime. For the induction step, we introduce the problem ORBIT COSET generalizing both HIDDEN TRANSLATION and HIDDEN SUBGROUP, and prove a powerful self-reducibility result: ORBIT COSET in a finite group  $G$ , is reducible to ORBIT COSET in  $G/N$  and subgroups of  $N$ , for any solvable normal subgroup  $N$  of  $G$ .

## 1 Introduction

Quantum computing is an extremely active research area (for surveys see e.g. [RP00, Aha98, Pre98, NC00]), where a growing trend is to cast quantum algorithms in a group theoretical setting. In this setting, we are given a finite group  $G$  and, besides the group operations, we also have at our disposal a function  $f$  mapping  $G$  into a finite set. The function  $f$  can be queried via an oracle. The complexity of an algorithm is measured by the overall running time counting one query as one computational step. The most important unifying problem of group theory for the purpose of quantum algorithms has turned out to be HIDDEN SUBGROUP, which can be cast in the following broad terms: Let  $H$  be a subgroup of  $G$  such that  $f$  is constant on each left coset of  $H$  and distinct on different left cosets. We say that  $f$  *hides* the subgroup  $H$ . The task is to determine the *hidden subgroup*  $H$ .

While no classical algorithm can solve this problem with polynomial query complexity, the biggest success of quantum computing until now is that it can be solved by a quantum algorithm efficiently whenever  $G$  is abelian. We will refer to this algorithm as the standard algorithm for HIDDEN SUBGROUP. The main tool for this solution is Fourier sampling based on the (approximate) quantum Fourier transform for abelian groups which can be efficiently implemented quantumly [Kit95]. Simon's xor-mask finding [Sim97], Shor's factorization and discrete logarithm finding algorithms [Sho97], and Kitaev's algorithm [Kit95] for the abelian stabilizer problem are all special cases of this general solution.

Addressing HIDDEN SUBGROUP in the non-abelian case is considered to be one of the most important challenge at present in quantum computing. Besides its intrinsic mathematical interest, the importance of this problem is enhanced by the fact that it contains as a special case the graph isomorphism problem. Unfortunately, non-abelian HIDDEN SUBGROUP seems to be much more difficult than the abelian case, and although considerable efforts were spent on it in the last years, only a few successes can be reported. They can be divided in two categories. The standard algorithm is extended to some non-abelian groups in [RB98, HRT00, GSVV01] using the quantum Fourier transform over these groups.

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Unfortunately, efficient quantum Fourier transform implementations are known only for a few non-abelian groups [Bea97, PRB99, RB98, HRT00]. In a different approach, HIDDEN SUBGROUP was efficiently solved in the context of specific black-box groups [BS84, Wat01] by [IMS01] without using the Fourier transform on the group.

In face of the apparent hardness of HIDDEN SUBGROUP in non-abelian groups, a natural line of research is to address subproblems of HIDDEN SUBGROUP which, in some groups, centralize the main difficulty of the original problem. In a pioneering paper, Ettinger and Høyer [EH00], in the case of dihedral groups, implicitly considered another paradigmatic group problem, HIDDEN TRANSLATION. Here we are given two injective functions  $f_0$  and  $f_1$  from a finite group  $G$  to some finite set such that, for some group element  $u$ , the equality  $f_1(xu) = f_0(x)$  holds for every  $x$ . The task is to find the *translation*  $u$ . In fact, whenever  $G$  is abelian, HIDDEN TRANSLATION is an instance of HIDDEN SUBGROUP in the semi-direct product  $G \rtimes \mathbb{Z}_2$ , where the hiding function is  $f(x, b) = f_b(x)$ . In that group  $f$  hides the subgroup  $H = \{(0, 0), (u, 1)\}$ . Actually, there is a quantum reduction also in the other direction and the two problems are quantum polynomial time equivalent [EH00]. A nice consequence of this equivalence is that instead of dealing with HIDDEN SUBGROUP in the non-abelian group  $G \rtimes \mathbb{Z}_2$ , we can address HIDDEN TRANSLATION in the abelian group  $G$ . Ettinger and Høyer [EH00] have shown that HIDDEN TRANSLATION can be solved by a two-step procedure when  $G = \mathbb{Z}_N$  is cyclic: polynomial number of Fourier samplings over the abelian group  $\mathbb{Z}_N \times \mathbb{Z}_2$  followed by an exponential classical stage without further queries.

Our first result (**Theorem 1**) is an efficient quantum algorithm for HIDDEN TRANSLATION in the case of elementary abelian  $p$ -groups, that is groups  $\mathbb{Z}_p^n$ , for any fixed prime number  $p$ . The quantum part of our algorithm is the same as in the Ettinger and Høyer procedure: it consists in performing Fourier sampling over the abelian group  $\mathbb{Z}_p^n \times \mathbb{Z}_2$ . But while their classical postprocessing requires exponential time, here we are able to recover classically the translation in polynomial time from the sampling. It turns out that Fourier sampling produces vectors  $y$ 's non-orthogonal to the translation  $u$ , that is we get linear inequations for the unknown  $u$ . This is different from the situation in the standard algorithm for the abelian HIDDEN SUBGROUP, where only vectors orthogonal to the hidden subgroup are generated. We show that, after a polynomial number of samplings, the system of linear inequations has a unique solution with high probability, which we are able to determine in deterministic polynomial time. An immediate consequence of Theorem 1 is that HIDDEN SUBGROUP is efficiently solvable by a quantum algorithm in  $\mathbb{Z}_p^n \rtimes \mathbb{Z}_2$ .

We remark that it is possible to extend the previous approach to solve HIDDEN TRANSLATION in the groups  $\mathbb{Z}_{p^k}^n$ , where  $p^k$  is a fixed prime power, but we do not know how to extend it to an arbitrary abelian group, even of bounded exponent. Therefore, we embark in a radically new direction whose basic idea is self-reducibility. Since HIDDEN TRANSLATION is not well-suited for this approach, we will consider ORBIT COSET which is a quantum generalization of both HIDDEN TRANSLATION and HIDDEN SUBGROUP. ORBIT COSET involves quantum group actions, that is groups acting on a finite set of mutually orthogonal quantum states. Given two such states  $|\phi_0\rangle$  and  $|\phi_1\rangle$ , the problem consists in finding their orbit coset, that is the stabilizer of  $|\phi_1\rangle$  and a group element that maps  $|\phi_1\rangle$  to  $|\phi_0\rangle$ .

With a slight modification, our algorithm of Theorem 1 also works for ORBIT COSET in  $\mathbb{Z}_p^n$  whenever many copies of the input states are given. Moreover we show that ORBIT COSET has the following self-reducibility property in any group  $G$ : it is reducible to ORBIT COSET in  $G/N$  and subgroups of  $N$ , for any solvable subgroup  $N \triangleleft G$  (**Theorem 3**). This is the first time that such a general self-reducibility result has been obtained for a problem incorporating HIDDEN SUBGROUP. It involves a new technique based on constructing the uniform superposition of the orbit of a given quantum state (ORBIT SUPERPOSITION). We show how this problem is related to ORBIT COSET (**Theorem 2**). The self-reducibility of ORBIT COSET combined with its solvability for  $\mathbb{Z}_p^n$  enables us to design an efficient quantum algorithm for ORBIT COSET in groups that we call smoothly solvable groups (**Theorem 4**). These groups include solvable groups of bounded exponent and bounded derived series; in particular, upper triangular matrix groups of bounded

dimension over finite fields. For the special case of STABILIZER (i.e. ORBIT COSET when  $|\phi_1\rangle = |\phi_0\rangle$ ), we get an efficient quantum algorithm for an even larger class of solvable groups viz. for solvable groups having a smoothly solvable commutator subgroup (**Theorem 5**). As an immediate consequence, we get efficient quantum algorithms for HIDDEN TRANSLATION and HIDDEN SUBGROUP for the same groups as ORBIT COSET and STABILIZER respectively.

## 2 Preliminaries

### 2.1 Group theory and quantum computation backgrounds

We say that a quantum algorithm solves a problem with error  $\varepsilon$  if for every input it produces an output whose distance from a correct one is at most  $\varepsilon$ . We say that a problem  $\mathcal{P}$  is *reducible* to a finite set of problems  $\{\mathcal{Q}_i : i \in I\}$  with *error expansion*  $c > 0$ , if whenever each problem  $\mathcal{Q}_i$  has a quantum polynomial time algorithm with error  $\varepsilon$ , problem  $\mathcal{P}$  has also one with error  $c\varepsilon$ . We say that a computational problem can be solved in quantum polynomial time if there exists a quantum polynomial time algorithm that outputs the required solution with exponentially small error.

Our results concern groups represented in the general framework of black-box groups [BS84, Wat01] with unique encoding. In this model, the elements of a finite group  $G$  are uniquely encoded by binary strings of length  $O(\log|G|)$  and the group operations are performed by an oracle (the black-box). The groups are assumed to be input by generators. In the case of an abelian group  $G$ , this implies also that we have at our disposal the decomposition of  $G$  into  $\mathbb{Z}_{p_1^{k_1}} \times \dots \times \mathbb{Z}_{p_m^{k_m}}$ , where  $p_i^{k_i}$  are prime powers [CM01]. We use the notation  $\langle X \rangle$  for the subgroup generated by a subset  $X$  of  $G$ . We denote by induction  $G^{(k+1)}$  the commutator  $(G^{(k)})'$  of  $G^{(k)}$ , where  $H' = \langle \{h^{-1}k^{-1}hk : h, k \in H\} \rangle$  for any subgroup  $H$ . Whenever  $G$  is solvable, the decomposition of  $G$  into its *derived series*  $G = G^{(0)} \triangleright G^{(1)} \triangleright \dots \triangleright G^{(m)} = \{1_G\}$  can be computed by a randomized procedure [BCF<sup>+</sup>95]. Using quantum procedures of [Wat01][IMS01, Theorem. 10], we can compute the cyclic decomposition of each abelian factor group, and thereby expand the derived series to a *composition series*, where factor groups are cyclic of prime order. We introduce a shorthand notation for specific solvable groups for which most of our results will apply. We say that an abelian group is *smoothly abelian* if it can be expressed as the direct product of a subgroup of bounded exponent and a subgroup of polylogarithmic size in the order of the group. A solvable group is *smoothly solvable* if its derived series is of bounded length and has smoothly abelian factor groups. For a smoothly solvable group  $G$ , by combining the procedures of [CM01, Wat01, IMS01], we can compute in quantum polynomial a *smooth series*  $G = G_0 \triangleright G_1 \triangleright \dots \triangleright G_m = \{1_G\}$ , where  $m$  is bounded, each factor group  $G_i/G_{i+1}$  is either elementary abelian of bounded exponent or abelian of size polylogarithmic in the order of  $G$ .

When  $G$  is abelian, we identify with  $G$  the set  $\widehat{G}$  of characters of  $G$  via some fixed isomorphism  $y \mapsto \chi_y$ . The *orthogonal* of  $H \leq G$  is defined as  $H^\perp = \{y \in G : \forall h \in H, \chi_y(h) = 1\}$ . The *quantum Fourier transform* over  $G$  is the unitary transformation defined for every  $x \in G$  by  $\text{QFT}_G|x\rangle = \frac{1}{\sqrt{|G|}} \sum_{y \in G} \chi_y(x)|y\rangle$ . For the sake of convenience, we will use the exact quantum Fourier transform in our algorithm. The actual implementation [Kit95] introduces only exponentially small errors.

The following well known quantum Fourier sampling algorithm will be used as a building block, where  $G$  is a finite abelian group,  $S$  is a finite set and  $f : G \rightarrow S$  is given by a quantum oracle. This algorithm is actually the main ingredient for solving HIDDEN SUBGROUP in abelian groups when the function  $f$  hides a subgroup  $H \leq G$ . In that case, **Fourier sampling** <sup>$f$</sup> ( $G$ ) generates the uniform distribution over  $H^\perp$ .

**Fourier sampling<sup>f</sup>( $G$ )**

1. Create zero-state  $|0\rangle_G|0\rangle_S$ .
2. Create uniform superposition on first register.
3. Query function  $f$ .
4. Compute  $\text{QFT}_G$  on first register.
5. Observe and then output the first register.

A function  $f : G \rightarrow \mathbb{C}^S$  is a *quantum function* if, for every  $x \in G$ , the vector  $|f(x)\rangle$  has unit norm, and, for every  $x, y \in G$ , the vectors  $|f(x)\rangle$  and  $|f(y)\rangle$  are either the same or orthogonal. We say that the quantum function  $f$  is *given* by a quantum oracle if we have at our disposal a unitary transformation  $U_f$  satisfying  $U_f|x\rangle_G|0\rangle_S = |x\rangle_G|f(x)\rangle_S$ , for every  $x \in G$ .

## 2.2 The problems

Here we define the problems we are dealing with.

Let  $G$  be a finite group and let  $f_0, f_1$  be two injective functions from  $G$  to some finite set  $S$ . The couple of functions  $(f_0, f_1)$  can equivalently be considered as a single function  $f : G \times \mathbb{Z}_2 \rightarrow S$  where by definition  $f(x, b) = f_b(x)$ . We will use  $f$  for  $(f_0, f_1)$  when it is convenient in the coming discussion. We call an element  $u \in G$  the *translation* of  $f$  if for every  $x \in G$ , we have  $f_1(xu) = f_0(x)$ .

### HIDDEN TRANSLATION

*Input:* A finite group  $G$  and two injective functions  $f_0, f_1$  from  $G$  to some finite set  $S$  such that  $f = (f_0, f_1)$  has a translation  $u \in G$ .

*Output:*  $u$ .

For a finite group  $G$  and a finite set  $\Gamma$  of mutually orthogonal quantum states, we consider group actions of  $G$  on  $\Gamma$ . By definition,  $\alpha : G \times \Gamma \rightarrow \Gamma$  is a *group action* if for every  $x \in G$  the quantum function  $\alpha_x : |\phi\rangle \mapsto |\alpha(x, |\phi\rangle)\rangle$  is a permutation over  $\Gamma$  such that the application  $x \mapsto \alpha_x$  is a group homomorphism. We extend  $\alpha$  linearly to superpositions over  $\Gamma$ . When the group action  $\alpha$  is fixed, we use the notation  $|x \cdot \phi\rangle$  for the state  $|\alpha(x, |\phi\rangle)\rangle$ . Having a group action  $\alpha$  at our disposal means having a quantum oracle realizing the unitary transformation  $|x\rangle|\phi\rangle \mapsto |x\rangle|x \cdot \phi\rangle$ . For any positive integer  $t$ , we denote by  $\alpha^t$  the group action of  $G$  on  $\Gamma^t = \{|\phi\rangle^{\otimes t} : |\phi\rangle \in \Gamma\}$  defined by  $\alpha^t(x, |\phi\rangle^{\otimes t}) = |x \cdot \phi\rangle^{\otimes t}$ . The group action  $\alpha^t$  is equivalent to  $\alpha$  from the algebraic point of view. We need this because we define problems below where the input superpositions cannot, in general, be cloned. In most cases we need to work with several disentangled copies of the input superpositions in order to achieve reasonable solutions. The notion  $\alpha^t$  is introduced in order to capture these situations. Observe that one can construct a quantum oracle for  $\alpha^t$  using  $t$  queries to a quantum oracle for  $\alpha$ .

The *stabilizer* of a state  $|\phi\rangle \in \Gamma$  is the subgroup  $G_{|\phi\rangle} = \{x \in G : |x \cdot \phi\rangle = |\phi\rangle\}$ . Given  $|\phi\rangle \in \Gamma$ , the problem **STABILIZER** consists in finding  $O(\log|G|)$  generators for the subgroup  $G_{|\phi\rangle}$ .

**Proposition 1.** *Let  $G$  be a finite abelian group and let  $\alpha$  be a group action of  $G$ . When  $t = \Omega(\log(|G|) \log(1/\varepsilon))$ , then **STABILIZER** in  $G$  for the group action  $\alpha^t$  can be solved in quantum time  $\text{poly}(\log|G|) \log(1/\varepsilon)$  with error  $\varepsilon$ .*

*Proof.* Let  $|\phi\rangle^{\otimes t}$  be the input of **STABILIZER**. Let  $f$  be the quantum function on  $G$  defined by  $|f(x)\rangle = |x \cdot \phi\rangle$ , for every  $x \in G$ . Observe that  $f$  is an instance of the natural extension of **HIDDEN SUBGROUP** to quantum functions and it hides the stabilizer  $G_{|\phi\rangle}$ .

The algorithm for **STABILIZER** is simply the standard algorithm for the abelian **HIDDEN SUBGROUP** with error  $\varepsilon$ . In this algorithm, every query is of the form  $|x\rangle_G|0\rangle_S$ . We simulate the  $i^{\text{th}}$  query  $|x\rangle_G|0\rangle_S$

using the  $i^{\text{th}}$  copy of  $|\phi\rangle$ . The second register of the query is swapped with  $|\phi\rangle$ , and then we let act  $x$  on it. We remark that the standard algorithm for abelian HIDDEN SUBGROUP outputs  $O(\log|G|)$  generators for the hidden subgroup. ■

The *orbit* of a state  $|\phi\rangle \in \Gamma$  is the subset  $G(|\phi\rangle) = \{|x \cdot \phi\rangle : x \in G\}$ . The *orbit coset* of two states  $|\phi_0\rangle$  and  $|\phi_1\rangle$  of  $\Gamma$  is the set  $\{u \in G : |u \cdot \phi_1\rangle = |\phi_0\rangle\}$ . The orbit coset of  $|\phi_0\rangle$  and  $|\phi_1\rangle$  is either empty or a left coset  $uG_{|\phi_1\rangle}$  (or equivalently a right coset  $G_{|\phi_0\rangle}u$ ), for some  $u \in G$ . If the latter case occurs,  $|\phi_0\rangle$  and  $|\phi_1\rangle$  have conjugated stabilizers:  $G_{|\phi_0\rangle} = uG_{|\phi_1\rangle}u^{-1}$ . ORBIT COSET is a generalization of STABILIZER:

#### ORBIT COSET

*Input:* A finite group  $G$  acting on a finite set  $\Gamma$  of mutually orthogonal quantum states, and two quantum states  $|\phi_0\rangle, |\phi_1\rangle \in \Gamma$ .

*Output:*  $\begin{cases} \text{reject, if } G(|\phi_0\rangle) \cap G(|\phi_1\rangle) = \emptyset; \\ u \in G \text{ s.t. } |u \cdot \phi_1\rangle = |\phi_0\rangle \text{ and } O(\log|G|) \text{ generators for } G_{|\phi_1\rangle}, \text{ otherwise.} \end{cases}$

For a function  $f$  on  $G$ , the *superposition* of  $f$  on  $G$  is  $|f\rangle = \frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle |f(g)\rangle$ , and for  $x \in G$ , the *x-translate* of  $f$  is the function  $x \cdot f : g \mapsto f(gx)$ . Let  $\Gamma(f) = \{|x \cdot f\rangle : x \in G\}$ . Then a group element  $x$  acts naturally on  $|f'\rangle \in \Gamma(f)$  by mapping it to the superposition  $|x \cdot f'\rangle$  of its  $x$ -translate. We call this group action the *translation action*. The mapping  $|x\rangle |f'\rangle \mapsto |x\rangle |x \cdot f'\rangle$  is realized by right multiplying the first register of  $|f'\rangle$  by  $x^{-1}$ .

**Proposition 2.** *Let  $G$  be a finite group and let  $t = \text{poly}(\log|G|)$ . Then HIDDEN TRANSLATION (resp. HIDDEN SUBGROUP) is reducible to ORBIT COSET (resp. STABILIZER) for the group action  $\tau^t$ , where  $\tau$  denotes the translation action. The error expansion is 1.*

*Proof.* Let  $f$  be an instance of HIDDEN SUBGROUP. Then the stabilizer of  $|f\rangle^{\otimes t}$  is the group hidden by  $f$ . Let  $(f_0, f_1)$  be an instance of HIDDEN TRANSLATION. Then the orbit coset of  $|f_0\rangle^{\otimes t}$  and  $|f_1\rangle^{\otimes t}$  is the translation of  $(f_0, f_1)$ . ■

Given  $|\phi\rangle \in \Gamma$ , the problem ORBIT SUPERPOSITION consists in realizing the uniform superposition  $|G \cdot \phi\rangle = \frac{1}{\sqrt{|G(|\phi\rangle)|}} \sum_{|\phi'\rangle \in G(|\phi\rangle)} |\phi'\rangle$ . Note that this superposition can be also written as  $\frac{1}{\sqrt{|G/G_{|\phi\rangle}|}} \sum_{x \in G/G_{|\phi\rangle}} |x \cdot \phi\rangle$ .

## 3 Hidden Translation

The main result of this section is that HIDDEN TRANSLATION can be solved in polynomial time by a quantum algorithm in the special case when  $G = \mathbb{Z}_p^n$  for any fixed prime number  $p$ . In this section we use the additive notation for the group operation and  $x \cdot y$  stands for the standard inner product for  $x, y \in \mathbb{Z}_p^n$ . When  $p = 2$ , there already exists a quantum polynomial time algorithm since it is just an instance of Simon's xor-mask finding [Sim97].

The quantum part of our algorithm consists of performing **Fourier sampling** over the abelian group  $\mathbb{Z}_p^n \times \mathbb{Z}_2$ . It turns out that from the samples we will only use elements of the form  $(y, 1)$ . The important property of these elements  $y$  is that they are not orthogonal to the hidden translation. Some properties of the distribution of the samples are stated for general abelian groups in the following lemma.

**Lemma 1.** *Let  $f = (f_0, f_1)$  be an instance of HIDDEN TRANSLATION in a finite abelian group  $G$  having a translation  $u \neq 0$ . Then **Fourier sampling**<sup>f</sup>( $G \times \mathbb{Z}_2$ ) outputs an element in  $G \times \{1\}$  with probability  $1/2$ . Moreover, the probability of sampling the element  $(y, 1)$  depends only on  $\chi_y(u)$ , and is 0 if  $y \in u^\perp$ .*

*Proof.* The state of the algorithm **Fourier sampling**<sup>f</sup>( $G \times \mathbb{Z}_2$ ) before the final observation is

$$\frac{1}{2|G|} \sum_{x \in G} \sum_{y \in G} \sum_{c=0,1} \chi_y(x) (1 + (-1)^c \chi_y(u)) |y\rangle |c\rangle |f_0(x)\rangle.$$

■

When  $G = \mathbb{Z}_p^n$ , the value  $\chi_y(u)$  depends only on the inner product  $y \cdot u$  over  $\mathbb{Z}_p$ , and  $y \in u^\perp$  exactly when  $y \cdot u = 0$ . Therefore every  $(y, 1)$  generated satisfies  $y \cdot u \neq 0$ . Thus the output distribution is different from the usual one obtained for the abelian HIDDEN SUBGROUP where only vectors orthogonal to the hidden subgroup are generated. We overcome the main obstacle, which is that we do not know the actual value of the inner product  $y \cdot u$ , by raising these inequations to the power  $p-1$ . They become a system of polynomial equations since  $a^{p-1} = 1$  for every non-zero  $a \in \mathbb{Z}_p$ . In general, solving systems of polynomial equations over any finite field is NP-complete. But using the other special feature of our distribution, which is that the probability of sampling  $(y, 1)$  depends only on the inner product  $y \cdot u$ , we are able to show that after a polynomial number of samplings, our system of equations has a unique solution with constant probability, and the solution can be determined in deterministic polynomial time.

To solve our system of polynomial equations, we linearize it in the  $(p-1)^{\text{st}}$  symmetric power of  $\mathbb{Z}_p^n$ . We think of  $\mathbb{Z}_p^n$  as an  $n$ -dimensional vector space over  $\mathbb{Z}_p$ . For a fixed prime number  $p$  and an integer  $k \geq 0$ , let  $\mathbb{Z}_p^{(k)}[x_1, \dots, x_n]$  be the  $k^{\text{th}}$  symmetric power of  $\mathbb{Z}_p^n$  which will be thought of as the vector space, over the finite field  $\mathbb{Z}_p$ , of homogeneous polynomials of degree  $k$  in variables  $x_1, \dots, x_n$ . The monomials of degree  $p-1$  form a basis of  $\mathbb{Z}_p^{(p-1)}[x_1, \dots, x_n]$ , whose dimension is therefore  $\binom{n+p-2}{p-1}$ , which is polynomial in  $n$ .

For every  $y = (a_1, \dots, a_n) \in \mathbb{Z}_p^n$ , we define  $y^{(k)} \in \mathbb{Z}_p^{(k)}[x_1, \dots, x_n]$  as the polynomial  $(\sum_{j=1}^n a_j x_j)^k$ . Now observe that if the hidden translation vector is  $u = (u_1, \dots, u_n)$  then the vector  $u^* \in \mathbb{Z}_p^{(k)}[x_1, \dots, x_n]$  which for every monomial  $x_1^{e_1} \cdots x_n^{e_n}$  has coordinate  $u_1^{e_1} \cdots u_n^{e_n}$ , satisfies  $y^{(p-1)} \cdot u^* = (y \cdot u)^{p-1}$ . Therefore each linear inequation  $y \cdot u \neq 0$  over  $\mathbb{Z}_p^n$  will be transformed into the linear equation  $y^{(p-1)} \cdot U = 1$  over  $\mathbb{Z}_p^{(p-1)}[x_1, \dots, x_n]$ , where  $U$  is a  $\dim \mathbb{Z}_p^{(p-1)}[x_1, \dots, x_n]$ -sized vector of unknowns.

In fact, the polynomials  $y^{(p-1)}$  have full rank when  $y$  ranges over  $\mathbb{Z}_p^n$ . Moreover, in what is the main part of our proof, we show in Lemma 3 that whenever the span of  $y^{(p-1)}$  for the samples  $y$  is not  $\mathbb{Z}_p^{(p-1)}[x_1, \dots, x_n]$ , our sampling process furnishes with constant probability a vector  $y \in \mathbb{Z}_p^n$  such that  $y^{(p-1)}$  is linearly independent from the  $y^{(p-1)}$  for the previously sampled  $y$ . This immediately implies that if our sample size is of the order of the dimension of  $\mathbb{Z}_p^{(p-1)}[x_1, \dots, x_n]$ , the polynomials  $y^{(p-1)}$  are of full rank with high probability. When the polynomials have full rank, the linear equations  $y^{(p-1)} \cdot U = 1$  have exactly one solution which is  $u^*$ . From this unique solution one can easily recover a vector  $v$  such that  $v^* = u^*$ . Since  $u$  is of the form  $av$ , for some  $0 < a < p$ , the translation vector can be found by checking the  $(p-1)$  possibilities.

The following combinatorial lemma is at the basis of the correctness of our procedure.

**Lemma 2 (Line Lemma).** *Let  $y, z \in \mathbb{Z}_p^n$  and let  $L_{z,y} = \{(z + ay)^{(p-1)} : 0 \leq a \leq p-1\}$ . Then  $y^{(p-1)} \in \text{Span}(L_{z,y})$ .*

*Proof.* Let  $M_{z,y} = \{z^{(k)} y^{(p-1-k)} : 0 \leq k \leq p-1\}$ . Clearly  $\text{Span}(L_{z,y})$  is included in  $\text{Span}(M_{z,y})$ . We claim that the inverse inclusion is also true since the determinant of  $L_{z,y}$  in  $M_{z,y}$  is non-zero. Indeed, it is  $\left(\prod_{k=0}^{p-1} \binom{p-1}{k}\right) V(0, 1, 2, \dots, p-1)$ , where  $V$  denotes the Vandermonde determinant. The lemma now follows because  $M_{z,y}$  contains  $y^{(p-1)}$ . ■

Since the proof of the following proposition uses similar ideas as the proof of the Line Lemma, it is omitted.

**Proposition 3.**  $\mathbb{Z}_p^{(p-1)}[x_1, \dots, x_n]$  is spanned by  $y^{(p-1)}$  as  $y$  ranges over  $\mathbb{Z}_p^n$ .

We are now ready to prove our main lemma.

**Lemma 3.** Let  $u \in \mathbb{Z}_p^n$ ,  $u \neq 0$  and  $W$  be a subspace of  $\mathbb{Z}_p^{(p-1)}[x_1, \dots, x_n]$ . We set  $R = \{y \in \mathbb{Z}_p^n : y^{(p-1)} \in W\}$ . For  $k = 0, \dots, p-1$ , let  $V_k = \{y \in \mathbb{Z}_p^n : y \cdot u = k\}$  and  $R_k = R \cap V_k$ . If  $W \neq \mathbb{Z}_p^{(p-1)}[x_1, \dots, x_n]$ , then  $|R_k|/|V_k| \leq (p-1)/p$  for  $k = 1, \dots, p-1$ .

*Proof.* Since  $W \neq \mathbb{Z}_p^{(p-1)}[x_1, \dots, x_n]$ , Proposition 3 implies that  $R \neq \mathbb{Z}_p^n$ . We consider two cases. In the first case,  $V_0 \subseteq R$ . This implies that  $R_1$  is a proper subset of  $V_1$ . Choose any  $y \in V_1 \setminus R_1$ . Then by Lemma 2, in every coset of  $\langle y \rangle$  there is an element outside of  $R$ . A coset of  $\langle y \rangle$  contains exactly one element from each  $V_k$ ,  $k = 0, \dots, p-1$ . Hence  $\cup_{k \neq 0} V_k$  is partitioned into equal parts, each part of size  $p-1$ , by intersecting with the cosets of  $\langle y \rangle$ . In each part, there is an element outside of  $R$ . Therefore  $|\cup_{k \neq 0} R_k|/|\cup_{k \neq 0} V_k| \leq (p-2)/(p-1)$ . Now observe that  $R_k = \{ky : y \in R_1\}$  for  $k = 1, \dots, p-1$ . Therefore the sets  $R_k$  have the same size, and the values  $|R_k|/|V_k|$  are the same for  $k = 1, \dots, p-1$ . Thus  $|R_k|/|V_k| \leq (p-2)/(p-1)$  for  $k = 1, \dots, p-1$ , and the statement follows.

In the second case,  $V_0 \not\subseteq R$ . Therefore, there is an element  $y \in V_0 \setminus R_0$ . Then every  $V_k$ ,  $k = 0, \dots, p-1$ , is a union of cosets of  $\langle y \rangle$ . The Line Lemma below implies that every coset of  $\langle y \rangle$  contains an element outside of  $R$ . This proves that  $|R_k|/|V_k| \leq (p-1)/p$  for  $k = 1, \dots, p-1$ . This completes the proof of the lemma. ■

We now specify the algorithm **Translation finding** and prove that, with high probability, it finds the hidden translation in quantum polynomial time.

**Translation finding**<sup>*f*</sup>( $\mathbb{Z}_p^n$ )

0. If  $f_0(0) = f_1(0)$  then output 0.
1.  $N \leftarrow 13p \binom{n+p-2}{p-1}$ .
2. For  $i = 1, \dots, N$  do  $(z_i, b_i) \leftarrow \text{Fourier sampling}^f(\mathbb{Z}_p^n \times \mathbb{Z}_2)$ .
3.  $\{y_1, \dots, y_M\} \leftarrow \{z_i : b_i = 1\}$ .
4. For  $i = 1, \dots, M$  do  $Y_i \leftarrow y_i^{(p-1)}$ .
5. Solve the system of linear equations  $Y_1 \cdot U = 1, \dots, Y_M \cdot U = 1$ .
6. If there are several solutions then abort.
7. Let  $1 \leq j \leq n$  be such that the coefficient of  $x_j^{p-1}$  is 1 in  $U$ .
8. Let  $v = (v_1, \dots, v_n) \in \mathbb{Z}_p^n$  be such that  $v_j = 1$  and  $v_k$  is the coefficient of  $x_k x_j^{p-2}$  in  $U$  for  $k \neq j$ .
9. Find  $0 < a < p$  such that  $f_0(0) = f_1(av)$ .
10. Output  $av$ .

**Theorem 1.** For every prime number  $p$ , every integer  $n \geq 1$ , and every function  $f$  having a translation in  $\mathbb{Z}_p^n$ , Algorithm **Translation Finding**<sup>*f*</sup>( $\mathbb{Z}_p^n$ ) aborts with probability less than  $1/2$ , and when it does not abort it outputs the translation of  $f$ . The query complexity of the algorithm is  $O(p(n+p)^{p-1})$ , and its time complexity is  $(n+p)^{O(p)}$ .

*Proof.* Because of Step 0 of the algorithm, we can suppose w.l.o.g. that the translation  $u$  of  $f$  is non-zero.

If the algorithm does not abort, then  $U = u^*$  is the unique solution of the system in Step 5. When the coefficient of  $x_j^{p-1}$  is 1 in  $U$ , then  $u_j \neq 0$ . Also,  $u_k = u_j v_k$  for every  $k$ . Thus,  $u = u_j v$  and  $u$  is found in Step 9 for  $a = u_j$ .

From Lemma 1, the probability that **Fourier sampling**<sup>*f*</sup>( $\mathbb{Z}_p^n \times \mathbb{Z}_2$ ) outputs  $(y, 1)$  for some  $y$  is  $1/2$ . Therefore the expected value of  $M$  is  $N/2$ , and  $M > N/3$  with probability  $1 - e^{-N/18} < 1/4$  because of

Chernoff bound. If the system  $Y_1, \dots, Y_M$  has full rank, then it has a unique solution. By Lemmas 1 and 3, the expected number of linear equations that guarantee that the system has full rank is  $p \binom{n+p-2}{p-1}$ . Since  $N/3 > 4p \binom{n+p-2}{p-1}$ , the solution  $U$  is unique with probability at least  $3/4$  using Markov's inequality. Thus, the total probability of aborting is less than  $1/2$ .  $\blacksquare$

**Corollary 1.** *Let  $p$  be a fixed prime. HIDDEN TRANSLATION in  $\mathbb{Z}_p^n$  can be solved in quantum polynomial time.*

*Proof.* We perform two modifications in Algorithm **Translation finding**. First, to get error  $\varepsilon$ , the integer  $N$  is multiplied by  $O(\log(1/\varepsilon))$ . Moreover, we assumed in the algorithm that there is an oracle for  $f = (f_0, f_1)$ , that is the functions  $f_0$  and  $f_1$  can be quantumly selected. This is not possible in general when  $f_0$  and  $f_1$  are given by two distinct oracles. Therefore we replace the oracle access  $|x\rangle|b\rangle|0\rangle_S \mapsto |x\rangle|b\rangle|f_b(x)\rangle$  by  $|x\rangle|b\rangle|0\rangle_S \mapsto |x\rangle|b\rangle|f_b(x)\rangle|f_{1-b}(-x)\rangle$ .

With this type of oracle access the algorithm **Translation finding** performs just as well.

Let us now show how to simulate this new oracle access. From  $|x\rangle|b\rangle|0\rangle_S|0\rangle_S$  we compute  $|(-1)^b x\rangle|b\rangle|0\rangle_S|0\rangle_S$ , and then we call  $f_0$  and get  $|(-1)^b x\rangle|b\rangle|f_0((-1)^b x)\rangle_S|0\rangle_S$ . We multiply the first register by  $(-1)$  and call  $f_1$  which gives  $|(-1)^{b+1} x\rangle|b\rangle|f_0((-1)^b x)\rangle_S|f_1((-1)^{b+1} x)\rangle_S$ . Finally, we multiply the first register by  $(-1)^{b+1}$ , and swap the last two registers when  $b = 1$ .  $\blacksquare$

Since there is a quantum reduction from HIDDEN SUBGROUP to HIDDEN TRANSLATION for  $\mathbb{Z}_p^n \rtimes \mathbb{Z}_2$  [EH00], we obtain the following corollary.

**Corollary 2.** *Let  $p$  be a fixed prime. HIDDEN SUBGROUP in  $\mathbb{Z}_p^n \rtimes \mathbb{Z}_2$  can be solved in quantum polynomial time.*

The algorithm **Translation finding** can also be extended to solve ORBIT COSET in  $\mathbb{Z}_p^n$ .

**Corollary 3.** *Let  $p$  be a prime. Let  $\alpha$  be a group action of  $\mathbb{Z}_p^n$ . When  $t = \Omega(p(n+p)^{p-1} \log(1/\varepsilon))$ , ORBIT COSET in  $\mathbb{Z}_p^n$  for  $\alpha^t$  can be solved in quantum time  $(n+p)^{O(p)} \log(1/\varepsilon)$  with error  $\varepsilon$ .*

*Proof.* Let  $(|\phi_0\rangle^{\otimes t}, |\phi_1\rangle^{\otimes t})$  be the input of ORBIT COSET. We can suppose w.l.o.g. that the stabilizers of  $|\phi_0\rangle$  and  $|\phi_1\rangle$  are trivial. Indeed the stabilizers can be computed by Proposition 1. If they are different then the algorithm obviously has to reject, otherwise we can work in the factor group  $\mathbb{Z}_p^n/G_{|\phi_0\rangle} = \mathbb{Z}_p^{n'}$ , for some  $n' \leq n$ .

For  $b = 0, 1$ , let  $f_b$  be the injective quantum function on  $G$  defined by  $|f_b(x)\rangle = |x \cdot \phi_b\rangle$ , for every  $x \in G$ . If the orbit coset of  $(|\phi_0\rangle, |\phi_1\rangle)$  is empty, then  $f_0$  and  $f_1$  have distinct ranges. Otherwise the orbit coset of  $(|\phi_0\rangle, |\phi_1\rangle)$  is a singleton  $\{u\}$ , and  $(f_0, f_1)$  have the translation  $u$ .

The algorithm for ORBIT COSET on input  $(|\phi_0\rangle^{\otimes t}, |\phi_1\rangle^{\otimes t})$  is the algorithm **Translation finding** on input  $f = (f_0, f_1)$  with a few modifications described below. The oracle access to  $f$  is modified in the same way as Corollary 1. We simulate the  $i^{\text{th}}$  query  $|x\rangle|b\rangle|0\rangle_S|0\rangle_S$  using the  $i^{\text{th}}$  copy of  $(|\phi_0\rangle, |\phi_1\rangle)$ . The last two registers are swapped with  $|\phi_b\rangle|\phi_{1-b}\rangle$ , and then we let act  $x$  on the  $|\phi_b\rangle$  and  $(-x)$  on  $|\phi_{1-b}\rangle$ .

The equality tests in steps 0 and 9 are replaced by the swap test [BCWW01, GC01] iterated  $O(\log(1/\varepsilon))$  times. Finally,  $N$  is multiplied by  $O(\log(1/\varepsilon))$ , and the algorithm rejects whenever the algorithm **Translation finding** aborts or there is no solution in steps 5 and 9.  $\blacksquare$

## 4 Orbit superposition

The purpose of this section is to show that ORBIT SUPERPOSITION is reducible to ORBIT COSET in solvable groups  $G$ . The proof will be by induction along a composition series of  $G$ . The induction step is based on

the technique of [Wat01] to create a uniform superposition of elements of  $G$ . One way of stating Watrous's result is that it solves ORBIT SUPERPOSITION for the case of the special action when  $G$  acts on itself by left multiplication. More precisely, the induction step uses the following lemma.

**Lemma 4.** *Let  $K$  be a finite group and  $\alpha$  be a group action of  $K$  on  $\Gamma$ . Let  $L \triangleleft K$  such that  $K/L$  is cyclic of prime order  $r$ , and  $|\phi\rangle \in \Gamma$ . Given an element  $z \in K - L$ , the number  $r$  and  $|\phi\rangle|L \cdot \phi\rangle^{\otimes t}$ , realizing  $|\phi\rangle|K \cdot \phi\rangle^{\otimes(t-1)}$ , is reducible to ORBIT COSET in  $K$  for  $\alpha$  with error expansion  $O(t)$ , for every positive integer  $t$ .*

*Proof.* The analysis of the algorithm will distinguish between two cases: case one is when  $K_{|\phi\rangle} \not\subseteq L$ , and case two is when  $K_{|\phi\rangle} \subseteq L$ . In the first case, for every  $x \in G$ ,  $|x \cdot (L \cdot \phi)\rangle = |K \cdot \phi\rangle$ , and in particular,  $|L \cdot \phi\rangle = |K \cdot \phi\rangle$ . In the second case,  $|K \cdot \phi\rangle = \frac{1}{\sqrt{r}} \sum_{i=0}^{r-1} |z^i \cdot (L \cdot \phi)\rangle$ , since the order  $r$  is prime.

The algorithm first computes  $t$  copies of  $\frac{1}{\sqrt{r}} \sum_{i=0}^{r-1} |i\rangle |z^i \cdot (L \cdot \phi)\rangle$ , from the  $t$  copies of  $|L \cdot \phi\rangle$ . We want to disentangle the first registers using Watrous's method. We apply the quantum Fourier transform over  $\mathbb{Z}_r$  in these registers. In the first case we obtain the state  $(|0\rangle|K \cdot \phi\rangle)^{\otimes t}$ , and in the second case we obtain the state  $(\frac{1}{\sqrt{r}} \sum_{j=0}^{r-1} |j\rangle |\psi_j\rangle)^{\otimes t}$ , where  $|\psi_j\rangle = \frac{1}{\sqrt{r}} \sum_{i=0}^{r-1} \omega_r^{ij} |z^i \cdot (L \cdot \phi)\rangle$ , and  $\omega_r$  is a fixed primitive  $r^{\text{th}}$ -root of unity.

We now describe the rest of the algorithm by specifying how it behaves on the terms of the above tensor products. Let  $|j_0\rangle|\psi_{j_0}\rangle|j_1\rangle|\psi_{j_1}\rangle \dots |j_{t-1}\rangle|\psi_{j_{t-1}}\rangle$  be such a term. If all the values  $j$  are 0 then the algorithm does nothing. Observe that if this happens, we already have  $t$  copies of the desired superposition  $|K \cdot \phi\rangle$ , independently of which case we are in. Otherwise, let  $j'$  be the first non-zero  $j$ . Note that this can only happen in case two. We swap  $|j_0\rangle|\psi_{j_0}\rangle$  and  $|j'\rangle|\psi_{j'}\rangle$ , and record the value  $j'$  in an ancilla register. For convenience of notation, we continue to refer to the first two registers as  $|j_0\rangle|\psi_{j_0}\rangle$ . Thus, we have ensured that  $j_0 \neq 0$ . Using  $|\psi_{j_0}\rangle$  our purpose will be to cancel the phases of all the other states  $|\psi_j\rangle$  for which  $j \neq 0$ . Observe that  $|l \cdot \psi_{j_0}\rangle = |\psi_{j_0}\rangle$  for every  $l \in L$  (and hence for every  $k \in K_{|\phi\rangle}$ ), and  $|z \cdot \psi_{j_0}\rangle = \omega_r^{-j_0} |\psi_{j_0}\rangle$ . Therefore if we set  $f = j(j_0)^{-1} \pmod{r}$  for some  $j \neq 0$ , then, for every  $i \in \{0, \dots, r-1\}$ ,  $l \in L$ , and  $k \in K_{|\phi\rangle}$ ,  $|(z^i l k)^f \cdot \psi_{j_0}\rangle = \omega_r^{-ij} |\psi_{j_0}\rangle$ .

We now complete the reduction by computing  $|\phi\rangle|\psi_{j_0}\rangle|K \cdot \phi\rangle$  from  $|\phi\rangle|\psi_{j_0}\rangle|\psi_j\rangle$ , when  $j \neq 0$ . Note that if  $j = 0$ ,  $|\psi_j\rangle$  is already equal to  $|K \cdot \phi\rangle$ . For every state  $|z^i l \cdot \phi\rangle$  of  $|\psi_j\rangle$ , we find the coset  $z^i l K_{|\phi\rangle}$  using ORBIT COSET in  $K$  for  $|z^i l \cdot \phi\rangle$  and  $|\phi\rangle$ . Let  $z^i l k$  be some representative of the coset where  $k \in K_{|\phi\rangle}$ . We let  $(z^i l k)^f$  act on  $|\psi_{j_0}\rangle$  and reverse the previous ORBIT COSET procedure. This realizes the transformation  $|\phi\rangle|\psi_{j_0}\rangle|z^i l \cdot \phi\rangle \mapsto \omega_r^{-ij} |\phi\rangle|\psi_{j_0}\rangle|z^i l \cdot \phi\rangle$ . The effect on  $|\phi\rangle|\psi_{j_0}\rangle|\psi_j\rangle$  is  $|\phi\rangle|\psi_{j_0}\rangle|K \cdot \phi\rangle$ . Since the first pair of registers remains unchanged, the process can be repeated for the other states, and therefore we get  $|\phi\rangle|j_0\rangle|\psi_{j_0}\rangle|j_1\rangle|K \cdot \phi\rangle \dots |j_{t-1}\rangle|K \cdot \phi\rangle$ , together with some garbage in the ancilla register. ■

**Theorem 2.** *Let  $G$  be a finite solvable group and let  $\alpha$  be a group action on  $\Gamma$ . Let  $|\phi\rangle \in \Gamma$ . Given  $|\phi\rangle^{\otimes(s+\lfloor \log|G| \rfloor + 1)}$ , realizing  $|\phi\rangle|G \cdot \phi\rangle^{\otimes s}$  is reducible to ORBIT COSET in subgroups of  $G$  for  $\alpha$  with error expansion  $O(s \log|G| + \log^2|G|)$ .*

*Proof.* Let us recall that the group  $G$  can be given with elements  $z_i$  and primes  $r_i$ , for  $i = 0, \dots, m-1$ , such that  $G$  has a composition series  $G = G_0 \triangleright G_1 \triangleright \dots \triangleright G_m = \{1_G\}$ , where  $G_i/G_{i+1}$  is cyclic of order  $r_i$  and is generated by  $z_i G_{i+1}$ . By induction, for  $i = m$  down to  $i = 0$ , we will produce the state  $|\phi\rangle|G_i \cdot \phi\rangle^{\otimes(s+i)}$ .

For  $i = m$ , by the hypothesis we have at least  $s + m + 1$  states  $|\phi\rangle = |G_m \cdot \phi\rangle$  since  $m \leq \log|G|$ . Assume now that we have  $|\phi\rangle$  and  $s + i$  copies of the state  $|G_i \cdot \phi\rangle$ . By applying Lemma 4 with  $K = G_{i-1}$ ,  $L = G_i$ ,  $z = z_{i-1}$  and  $r = r_{i-1}$ , we get  $s + i - 1$  copies of the state  $|G_{i-1} \cdot \phi\rangle$ . When  $i = 0$ , we obtain  $|\phi\rangle|G \cdot \phi\rangle^{\otimes s}$ . ■

## 5 Orbit coset self-reducibility

This section is based on the following theorem stating the reducibility of ORBIT COSET in  $G$  to ORBIT COSET in proper normal subgroups of  $G$  under some conditions. Given a group action  $\alpha$  of  $G$  on a finite set  $\Gamma$  of mutually orthogonal quantum states, we define for every proper normal subgroup  $N \triangleleft G$  the group action  $\alpha_N$  of  $G/N$  on  $\{|N \cdot \phi\rangle : |\phi\rangle \in \Gamma\}$  by  $\alpha_N(xN, |N \cdot \phi\rangle) = |x \cdot (N \cdot \phi)\rangle$ , for every  $x \in G$  and  $|\phi\rangle \in \Gamma$ . Note that this action is independent of the coset representative chosen.

**Theorem 3.** *Let  $G$  be a finite group and let  $N \triangleleft G$ ,  $N \neq G$  be solvable such that  $G$ ,  $N$  and  $G/N$  are black-box groups with unique encoding. Let  $\alpha$  be a group action of  $G$  and let  $s \geq 1$  be an integer. When  $t = \Omega(s + \log|G|)$ , ORBIT COSET (resp. STABILIZER) in  $G$  for  $\alpha^t$  is reducible to ORBIT COSET in subgroups of  $N$  for  $\alpha$  and ORBIT COSET (resp. STABILIZER) in  $G/N$  for  $(\alpha_N)^s$  with error expansion  $O(s \log|G| + \log^2|G|)$ .*

*Proof.* We first prove the statement for the STABILIZER reduction. The proof for the ORBIT COSET reduction uses the result for STABILIZER. This is indeed legitimate since STABILIZER is the special case of ORBIT COSET when the two inputs are identical.

Let  $|\phi\rangle^{\otimes t}$  be an instance of STABILIZER. Its stabilizer  $H$  is the same as the stabilizer of  $|\phi\rangle$ . First we compute  $O(\log|N|)$  generators for the intersection  $H_0 = H \cap N$  using STABILIZER in  $N$  for  $\alpha$  in quantum polynomial time. Then we use ORBIT COSET in  $N$  to construct  $H_1 \leq G$  which in fact will turn out to be  $H$ . The properties which will ensure that equality are  $H_0 \leq H_1 \leq H$  and  $H_1 N/N = H N/N$ . Indeed, the first property clearly implies that  $H_1 \cap N = H \cap N$ , which together with the second one gives that  $H_1 = H$  from the isomorphism theorem.

To construct  $H_1$  we add to  $H_0$  generators in  $H$  of  $HN/N$ . The construction proceeds in two steps. First, we find a set  $V \subseteq G$  which, when its elements are considered as coset representatives, contains a generator set for  $HN/N$ . Then, for every coset  $zN$  where  $z \in V$ , we find a coset representative of  $zN$  in  $H$ . This step is achieved via a reduction to ORBIT COSET in  $N$ . The collection of those representatives and  $H_0$  together generate the desired subgroup  $H_1$ .

The stabilizer of  $|N \cdot \phi\rangle$  for  $\alpha_N$  in  $G/N$  is  $HN/N$ . Therefore finding  $V$  is reducible to STABILIZER in  $G/N$  for  $(\alpha_N)^s$  on input  $|N \cdot \phi\rangle^{\otimes s}$ . By Theorem 2, creating this input is also reducible to ORBIT COSET in subgroups of  $N$  for  $\alpha$  on input  $s + \lfloor \log|G| \rfloor + 1$  copies of  $|\phi\rangle$ . Note that the size of  $V$  is  $O(\log|G/N|)$ .

We describe now how to find, using ORBIT COSET in  $N$ , for each  $z \in V$ , an element  $n \in N$  such that  $zn \in H$ . Fix  $z \in V$ . We can construct  $|\phi'\rangle = |z^{-1} \cdot \phi\rangle$  using a copy of  $|\phi\rangle$ . In the subgroup  $N$ , the states  $|\phi'\rangle$  and  $|\phi\rangle$  have the orbit coset  $nH_0$ . Thus the coset  $nH_0$  can be found using ORBIT COSET in  $N$  for  $\alpha$ .

We now turn to the proof of the ORBIT COSET reduction. Let  $(|\phi_0\rangle^{\otimes t}, |\phi_1\rangle^{\otimes t})$  be the input of ORBIT COSET. Their orbit coset is identical to the orbit coset of  $(|\phi_0\rangle, |\phi_1\rangle)$ , and it is either empty or  $uG_{|\phi_1\rangle}$ , for some  $u \in G$ . We compute  $H = G_{|\phi_1\rangle}$  using the above construction. When the orbit coset of the input is empty, the states  $|N \cdot \phi_0\rangle^{\otimes s}$  and  $|N \cdot \phi_1\rangle^{\otimes s}$  have also empty orbit coset. Otherwise they have the orbit coset  $uHN/N$ .

By Theorem 2, the constructions of states  $|N \cdot \phi_0\rangle^{\otimes s}$  and  $|N \cdot \phi_1\rangle^{\otimes s}$  are reducible to ORBIT COSET in  $N$  for  $\alpha$  on inputs  $s + \lfloor \log|G| \rfloor + 1$  copies of  $|\phi_0\rangle$  and  $|\phi_1\rangle$ . Then using ORBIT COSET in  $G/N$  for  $(\alpha_N)^s$  in input  $|N \cdot \phi_0\rangle^{\otimes s}$  and  $|N \cdot \phi_1\rangle^{\otimes s}$ , we reject if the inputs have empty orbit coset, or we find the coset  $(uHN)/N$ , that is an element  $v \in uHN$ .

Using ORBIT COSET in  $N$ , we can find an element  $n \in N$  such that  $vn \in uH$  by the method already used in the STABILIZER reduction. We construct the state  $|\phi'_0\rangle = |v^{-1} \cdot \phi_0\rangle$  using one copy of  $|\phi_0\rangle$ . Let us denote  $H_0 = H \cap N$ . Since in the subgroup  $N$ , the states  $|\phi'_0\rangle$  and  $|\phi_1\rangle$  have the orbit coset  $nH_0$ , where  $n \in N$  is such that  $vn \in uH$ , we complete the proof using ORBIT COSET in  $N$ .  $\blacksquare$

**Theorem 4.** *Let  $G$  be a smoothly solvable group and let  $\alpha$  be a group action of  $G$ . When  $t = (\log^{\Omega(1)}|G|) \log(1/\varepsilon)$ , ORBIT COSET can be solved in  $G$  for  $\alpha^t$  in quantum time  $\text{poly}(\log|G|) \log(1/\varepsilon)$  with error  $\varepsilon$ .*

*Proof.* As  $G$  is smoothly solvable, it has a smooth series  $G = G_0 \triangleright G_1 \triangleright \dots \triangleright G_{m-1} \triangleright G_m = \{1_G\}$ , where  $m$  is bounded,  $G_i/G_{i+1}$  is either elementary abelian of bounded exponent or of size polylogarithmic in the order of  $G$ . Observe that we have a cyclic prime power decomposition of each factor group  $G_i/G_{i+1}$ , and for this representation, we have a black-box oracle for the group action of  $G_i/G_{i+1}$  on  $\{|G_{i+1} \cdot \phi\rangle : |\phi\rangle \in \Gamma\}$ .

The proof is by induction on  $m$ . The case  $m = 0$  is trivial. For the induction, we can efficiently solve ORBIT COSET in the factor group  $G_0/G_1$ : if it is of polylogarithmic size we just do an exhaustive search, otherwise we apply Corollary 3. Therefore Theorem 3 reduces ORBIT COSET in  $G$  to ORBIT COSET in subgroups of  $G_1$ . Any subgroup  $K$  of  $G_1$  has a smooth series of length at most  $m-1$ , since the intersection of a smooth series for  $G_1$  with  $K$  gives a smooth series for  $K$ . The running time of the overall procedure is  $(\log|G|)^{O(m)} \log(1/\varepsilon)$ .  $\blacksquare$

**Theorem 5.** *Let  $G$  be a finite solvable group having a smoothly solvable commutator and let  $\alpha$  be a group action of  $G$ . When  $t = (\log^{\Omega(1)}|G|) \log(1/\varepsilon)$ , STABILIZER can be solved in  $G$  for  $\alpha^t$  in quantum time  $\text{poly}(\log|G|) \log(1/\varepsilon)$  with error  $\varepsilon$ .*

*Proof.* By Theorem 3, STABILIZER in  $G$  is reducible to STABILIZER in  $G/G'$  and ORBIT COSET in subgroups of  $G'$ . The factor group  $G/G'$  is abelian and subgroups of  $G'$  are smoothly solvable. Therefore, from Proposition 1 and Theorem 4 the statement follows.  $\blacksquare$

Since, by Proposition 2, HIDDEN TRANSLATION and STABILIZER are respectively reducible to ORBIT COSET and STABILIZER, we get similar results for these two problems.

**Corollary 4.** *HIDDEN TRANSLATION can be solved in smoothly solvable groups in quantum polynomial time. HIDDEN SUBGROUP can be solved in solvable groups having a smoothly solvable commutator subgroup in quantum polynomial time.*

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