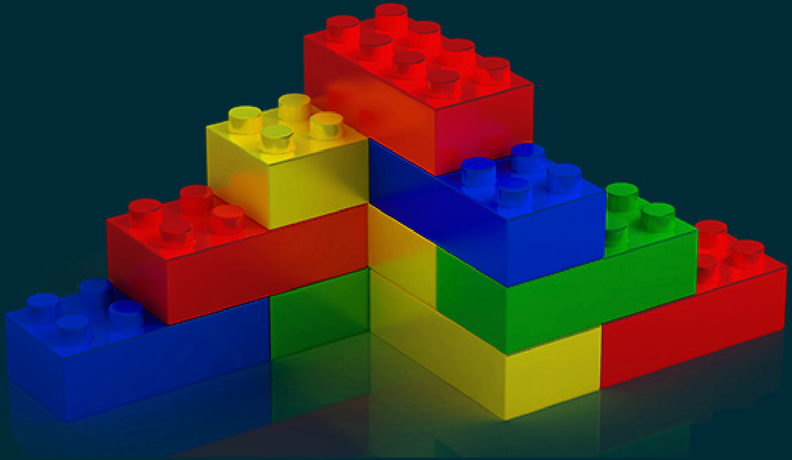
Quantum singular value transformation

András Gilyén

ELTE kvantum-infó szeminárium 2020 október 1

Quantum algorithm design



Many quantum algorithms have a common structure!

Szegedy quantum walk

Discrete-time Markov-chain on a weighted graph

Transition probability in one step (stochastic matrix)

$$P_{vu} = \Pr(\text{step to } v \,|\, \text{being at } u) = \frac{w_{vu}}{\sum_{v' \in U} w_{v'u}}$$

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$$W' := U^\dagger \cdot \mathrm{SWAP} \cdot U$$

$$W := U^\dagger \cdot \mathrm{SWAP} \cdot U((2|0\rangle\langle 0| \otimes I) - I)$$

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$[T_k(x) = \cos(k \arccos(x))$ Chebyshev polynomials: $T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)]$

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Proof: Proceed by induction, observe $T_0(P) = I$ ✓, $T_1(P) = P$ ✓

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Grover search and amplitude amplification

Amplitude amplification problem

Given U such that

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$$\pm \cos((2k+1)\arccos(\sqrt{p})) = \pm T_{2k+1}(\sqrt{p}).$$

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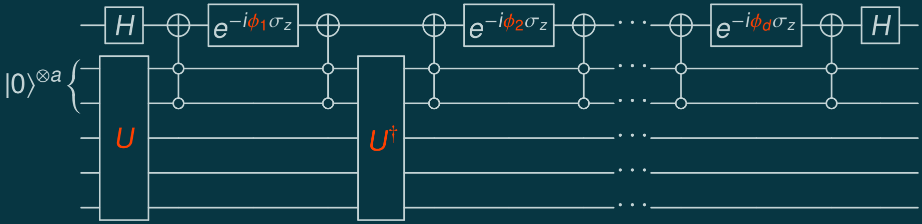

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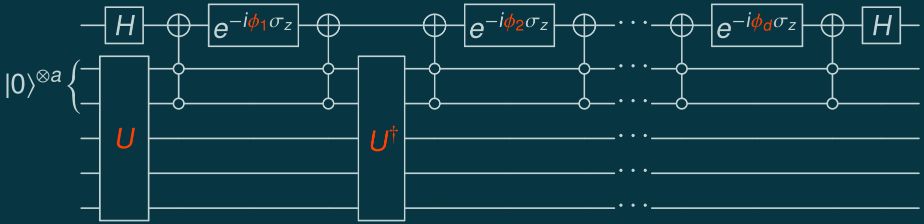

Simmilar result holds for even polynomials.

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Outline

Motivating example - the HHL algorithm

We want to solve large systems of linear equations

$$Ax = b.$$

A quantum computer can nicely work with exponential sized matrices!
Given $|b\rangle$, we can prepare a solution $\propto A^{-1}|b\rangle$.

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Matrix arithmetic on a quantum computer using block-encoding

$$\text{Target}: \; A; \; \text{Implementation}: \; U = \begin{bmatrix} A & \cdot \\ \cdot & \cdot \end{bmatrix}; \; \text{Algorithm}: \; U' = \begin{bmatrix} f(A) & \cdot \\ \cdot & \cdot \end{bmatrix}.$$

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Applications

- *Optimal Hamiltonian simulation [Low et al.], Quantum walks [Szegedy]*
- *Fixed point [Yoder et al.] and Oblivious ampl. ampl. [Berry et al.]*
- *HHL, Regression [Chakraborty et al.], ML [Kerendis & Prakash], Property testing, . . .*

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A way to represent large matrices on a quantum computer efficiently

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- a POVM operator M given we can sample from the rand.var.: $\mathrm{Tr}(\rho M)$,

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In particular if $(\langle 0| \otimes I)U_i(|0\rangle \otimes I) = A_i$, then it is a block-encoding of

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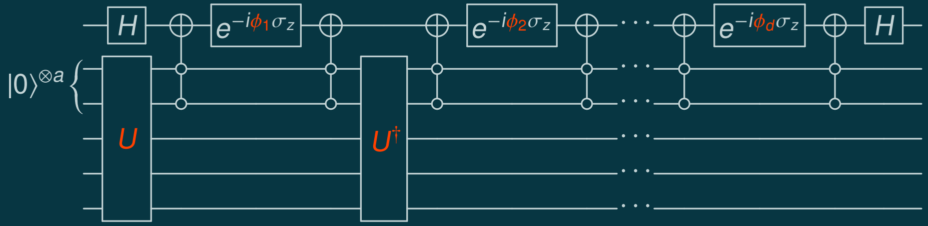

Quantum Singular Value Transformation (QSVT)

Our main theorem about QSVT

Let $P\colon [-1,1] \to [-1,1]$ be a degree-d odd polynomial map. Suppose that

$$U = \begin{bmatrix} A & . \\ . & . \end{bmatrix} = \begin{bmatrix} \sum_i \varsigma_i |w_i\rangle\langle v_i| & . \\ . & . \end{bmatrix} \Longrightarrow U_\Phi = \begin{bmatrix} \sum_i P(\varsigma_i) |w_i\rangle\langle v_i| & . \\ . & . \end{bmatrix},$$

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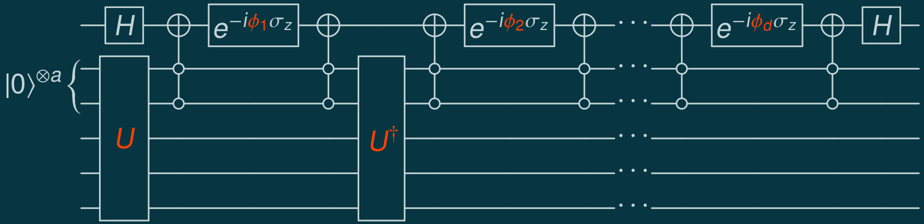

Simmilar result holds for even polynomials.

Quantum walks and Hermitian matrices

Connection to Szegedy quantum walks

Szegedy defined (2004) quantisation of a symmetric Markov chain M via a product of two reflection operators.

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$$\text{Markov chain}: \; M; \;\; \text{Updates}: \; W' = \begin{bmatrix} M & \cdot \\ \cdot & \cdot \end{bmatrix}; \;\; \text{Walk}: \; W^n = \begin{bmatrix} T_n(M) & \cdot \\ \cdot & \cdot \end{bmatrix}.$$

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$$U' = \begin{bmatrix} M^t & \cdot \\ \cdot & \cdot \end{bmatrix}.$$

Proof: x^t can be ε-apx. on $[-1, 1]$ with a degree-$\sqrt{2t \ln(2/\varepsilon)}$ polynomial.

The special case of Hermitian matrices

Singular value transf. = eigenvalue transf. [Low & Chuang (2017)]

Let $P\colon [-1,1] \to [-1,1]$ be a degree-d even/odd polynomial map.

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Removing parity constraint for Hermitian matrices

Let $P\colon [-1,1] \to [-\frac{1}{2}, \frac{1}{2}]$ be a degree-d polynomial map. Suppose that U is an a-qubit block-encoding of a Hermitian matrix H.

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$$U' = \begin{bmatrix} P(H) & . \\ . & . \end{bmatrix},$$

using d times U and $U^\dagger$, 1 controlled U, and $\mathcal{O}(ad)$ extra two-qubit gates.

Quantum signal processing & proof sketch of QSVT

Single qubit quantum control using σ_z phases?

Quantum signal processing & proof sketch of QSVT

Single qubit quantum control using σ_z phases?

$$R(x) := \begin{bmatrix} x & \sqrt{1-x^2} \\ \sqrt{1-x^2} & -x \end{bmatrix}; \quad e^{i\phi_0\sigma_z}R(x)e^{i\phi_1\sigma_z}\cdot\ldots\cdot R(x)e^{i\phi_d\sigma_z} = (*)?$$

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Theorem: Basic characterization [Low, Yoder, Chuang (2016)]

Let $d \in \mathbb{N}$; for all $\Phi \in \mathbb{R}^{d+1}$ we have

$$(*) = \begin{bmatrix} P_{\mathbb{C}}(x) & iQ_{\mathbb{C}}(x)\sqrt{1-x^2} \\ iQ^*_{\mathbb{C}}(x)\sqrt{1-x^2} & P^*_{\mathbb{C}}(x) \end{bmatrix},$$

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(iii) $\forall x \in [-1,1]$: $|P_{\mathbb{C}}(x)|^2 + (1-x^2)|Q_{\mathbb{C}}(x)|^2 = 1$.

Real quantum signal processing

Theorem: Focusing on the real part [Low, Yoder, Chuang (2016)]

Let $d \in \mathbb{N}$, and $P \in \mathbb{R}[x]$ be of degree d. There exists $\Phi \in \mathbb{R}^d$ such that

$$\prod_{j=1}^{d} \left(R(x) e^{i\phi_j \sigma_z} \right) = \begin{bmatrix} P_{\mathbb{C}}(x) & \cdot \\ \cdot & \cdot \end{bmatrix},$$

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Implementing the real part of a polynomial map

Direct implementation

$$-\boxed{e^{i\phi_d\sigma_z}}-\boxed{R(x)}-\boxed{e^{i\phi_{d-1}\sigma_z}}-\cdots-\boxed{R(x)}-\boxed{e^{i\phi_0\sigma_z}}- = \begin{bmatrix} P_{\mathbb{C}}(x) & \cdot \\ \cdot & \cdot \end{bmatrix}$$

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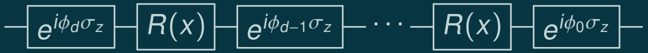

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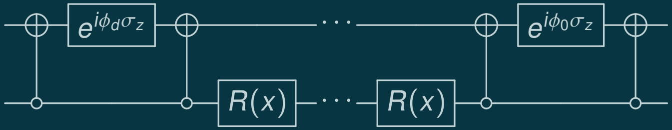

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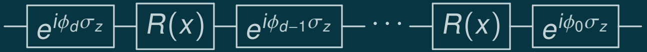

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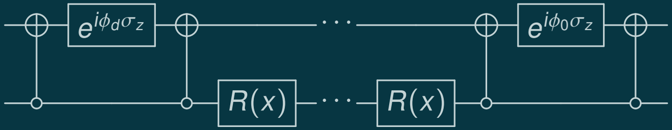

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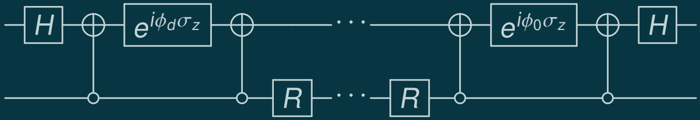

$$= \begin{bmatrix} \Re[P_{\mathbb{C}}] & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Generalisation to higher dimensions

1×1 case

$$\text{Input:} \begin{bmatrix} x & \cdot \\ \cdot & \cdot \end{bmatrix} \quad \text{Modulation:} \begin{bmatrix} e^{i\phi} & \\ & e^{-i\phi} \end{bmatrix} \quad \text{Output:} \begin{bmatrix} P(x) & \cdot \\ \cdot & \cdot \end{bmatrix}$$

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2×2 case (higher-dimensional case is similar)

Input unitary	Modulation	Output circuit
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Direct implementation of HHL / the pseudoinverse

Singular value decomposition and pseudoinverse

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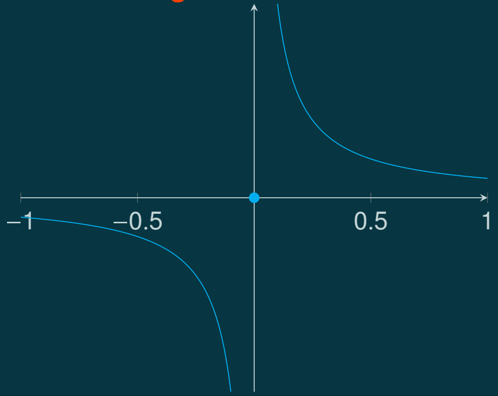

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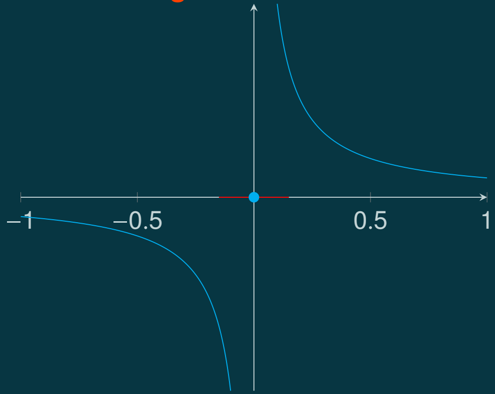

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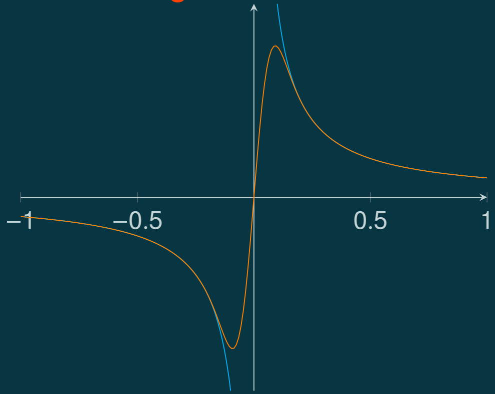

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Singular vector transformation and projection

New result: Singular vector transformation

Given a unitary U, and projectors $\widetilde{\Pi}, \Pi$, such that

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$$|\psi\rangle = \sum_{i=i}^{k} \alpha_i |\psi_i\rangle \quad \text{to} \quad |\phi\rangle = \sum_{i=i}^{k} \alpha_i |\phi_i\rangle.$$

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If $\varsigma_i \geq \delta$ for all $0 \neq \alpha_i$, we can ε-apx. using QSVT with compl. $\mathcal{O}\left(\frac{1}{\delta}\log\left(\frac{1}{\varepsilon}\right)\right)$.

Singular vector transformation and projection

New result: Singular vector transformation

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Amplitude amplification problem: Given U such that

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Note that $(|0\rangle\langle 0| \otimes I)U(|\psi_{\text{in}}\rangle\langle\psi_{\text{in}}|) = \sqrt{p}|0,\psi_{\text{good}}\rangle\langle\psi_{\text{in}}|$; we can apply QSVT.

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Given $t, \varepsilon > 0$, implement a unitary U', which is ε close to e^{itH}. Can be achieved with query complexity

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Approximate to ε-precision $\sin(tx)$ and $\cos(tx)$ with polynomials of degree as above. Then use QSVT and combine even/odd parts.

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Purified access $U_\rho : |0\rangle \mapsto \sum_i \sqrt{p_i}|\phi_i\rangle|\psi_i\rangle$, where $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$

An intuitive lower bound

Lower bound on eigenvalue transformation

Suppose that U is a block-encoding of a Hermitian matrix H from a family of operators. Let $f\colon [-1,1] \to \mathbb{C}$, then implementing a block-encoding of $f(H)$ requires at least $\left\| \frac{df}{dx} \right\|_I$ uses of U, if $I \subseteq [-\frac{1}{2}, \frac{1}{2}]$ is an interval of potential eigenvalues of H.

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The proof is based on an elementary argument about distinguishability of unitary operators.

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The proof is based on an elementary argument about distinguishability of unitary operators.

Optimality of pseudoinverse implementation

$$\text{Let } I := \left[\frac{1}{\kappa}, \frac{1}{2}\right] \text{ and let } f(x) := \frac{1}{\kappa x}, \text{ then } \left.\frac{df}{dx}\right|_{\frac{1}{\kappa}} = -\kappa.$$

Thus our implementation is optimal up to the $\log(1/\varepsilon)$ factor.

Summarizing the various speed-ups

Speed-up	Source of speed-up	Examples of algorithms
Exponential	Dimensionality of the Hilbert space	Hamiltonian simulation
	Precise polynomial approximations	Improved HHL algorithm
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Some more applications

- Quantum walks, fast QMA amplification, fast quantum OR lemma
- Quantum Machine learning: PCA, principal component regression
- “Non-commutative measurements” (for ground state preparation)
- Fractional queries
- $\vdots$