Efficient simulation of quantum circuits with Fermionic Linear Optics

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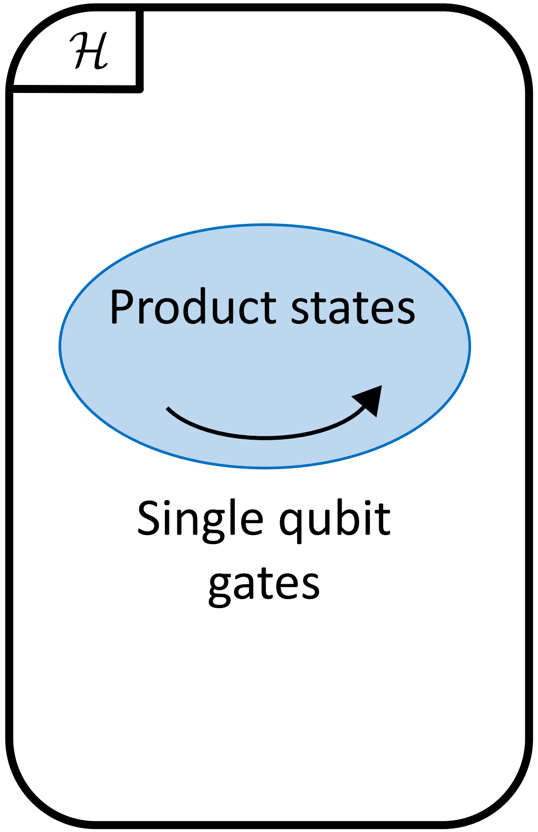
QuSZIT seminar, Budapest, 2023.12.04.

Classical simulation of quantum circuits

Classical simulation of large quantum circuits is exponentially slow in general, but there are some exceptions:

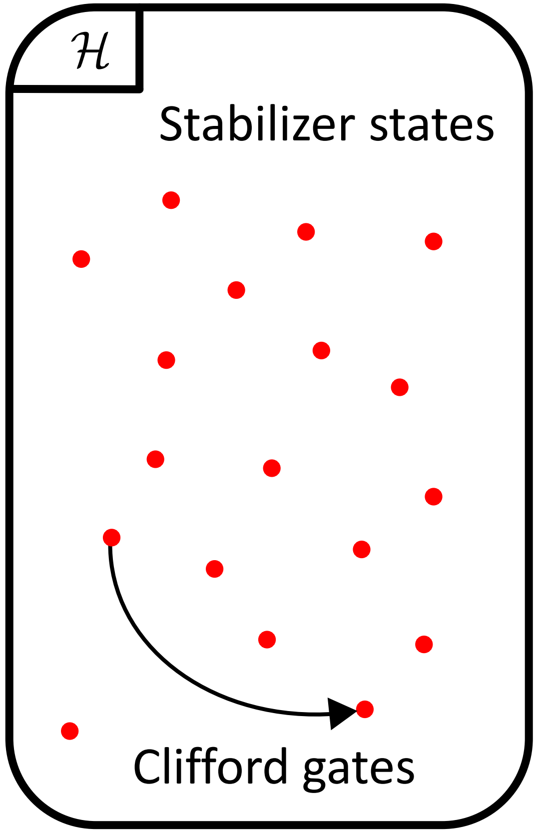
Non-entangled states



Product states

Single qubit gates

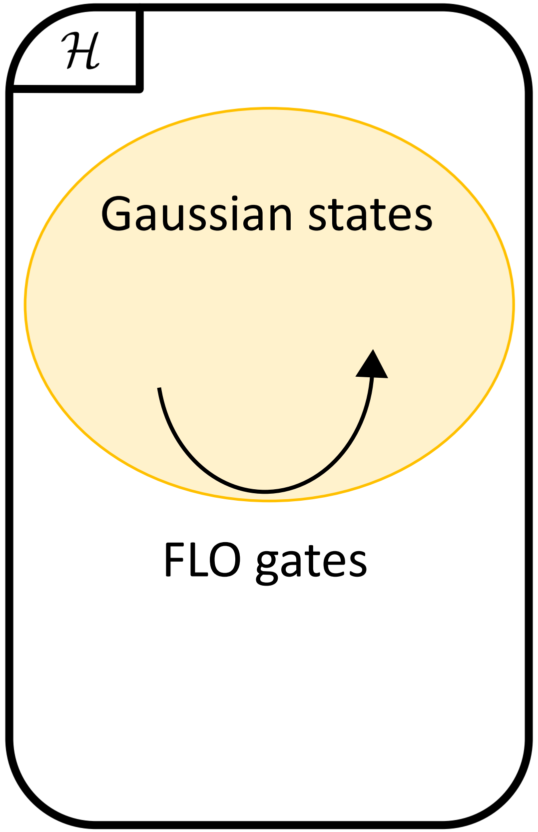
Clifford circuits



Stabilizer states

Clifford gates

Fermionic Linear Optics (FLO)



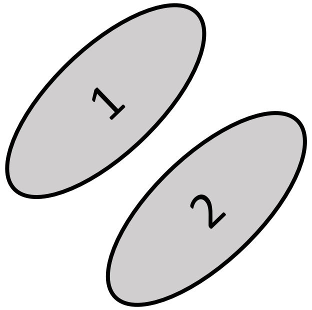
Gaussian states

FLO gates

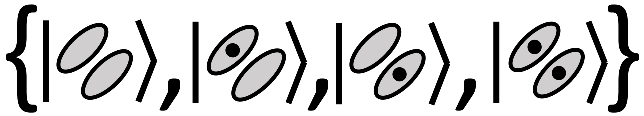
From qubits to fermions

C4 code – A fermionic system that realizes a qubit (Abrikosov representation)

Fermions in a double quantum dot:

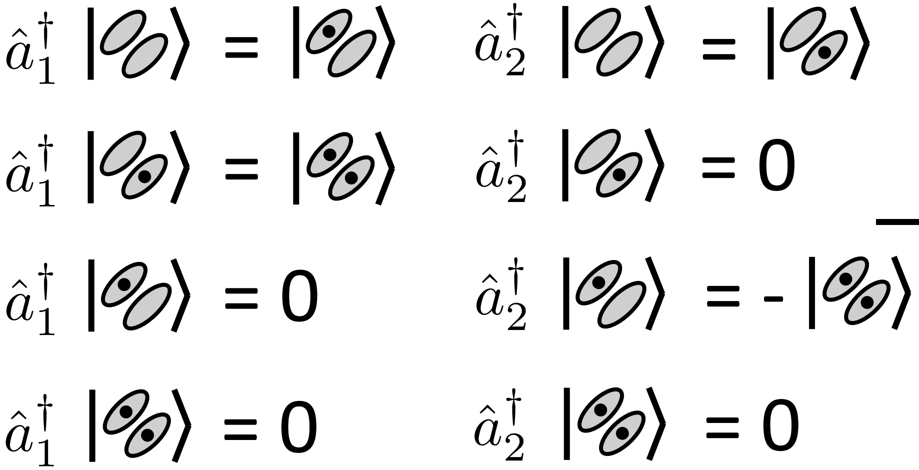


Basis:



Qubit subspace ($\mathcal{Q}$):

Fermionic operators:



$\hat{a}_1, \hat{a}_2$

Canonical anticommutation relations:

$$\{\hat{a}_j, \hat{a}_k^\dagger\} = \delta_{jk}$$

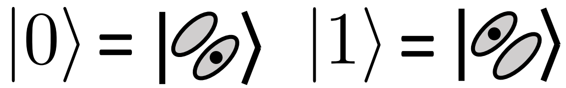
$$\{\hat{a}_j, \hat{a}_k\} = \{\hat{a}_j^\dagger, \hat{a}_k^\dagger\} = 0$$

C4 code stabilizer:

$$\hat{S}_{C4} = -(1 - 2\hat{a}_1^\dagger \hat{a}_1)(1 - 2\hat{a}_2^\dagger \hat{a}_2)$$

$$|\psi\rangle \in \mathcal{Q} : \quad \hat{S}_{C4}|\psi\rangle = |\psi\rangle$$

Qubit states:



Pauli operators: $\hat{Z} = 1 - 2\hat{a}_1^\dagger \hat{a}_1$ $\quad \hat{X} = \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1$

From fermions to Majoranas

Majorana fermions:

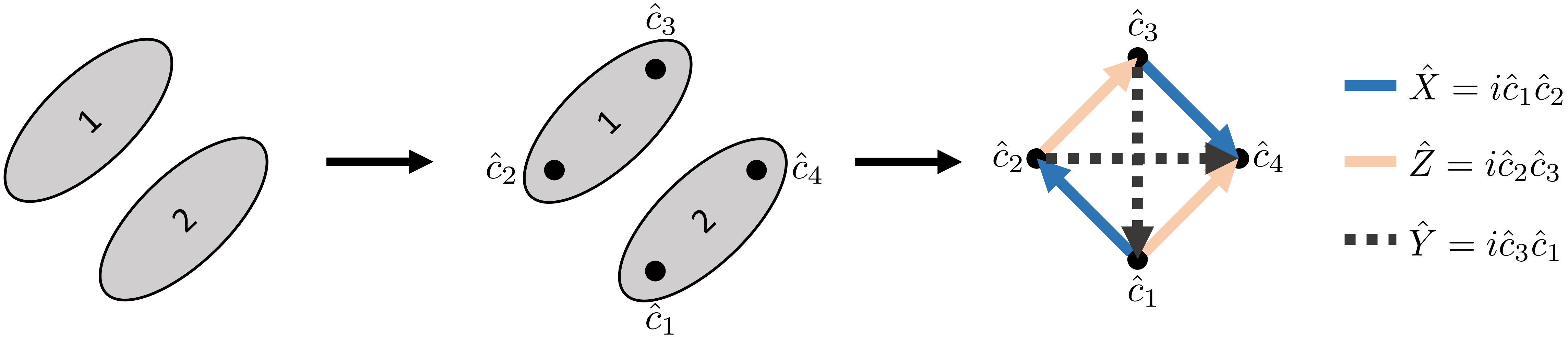
$$\hat{c}_1 = i(\hat{a}_2 - \hat{a}_2^\dagger) \qquad \hat{c}_2 = \hat{a}_1 + \hat{a}_1^\dagger$$

$$\hat{c}_3 = i(\hat{a}_1 - \hat{a}_1^\dagger) \qquad \hat{c}_4 = \hat{a}_2 + \hat{a}_2^\dagger$$

”Real” fermions

$$\hat{c}_j^\dagger = \hat{c}_j$$

$$\{\hat{c}_j, \hat{c}_k\} = 2\delta_{jk}$$



Pauli operators:

$$\hat{Z} = i\hat{c}_2\hat{c}_3 = 1 - 2\hat{a}_1^\dagger\hat{a}_1 - \cancel{\hat{a}_1^2} + \cancel{\hat{a}_1^{\dagger 2}}$$

$$\hat{X} = i\hat{c}_1\hat{c}_2 = \hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1 + \cancel{\hat{a}_1\hat{a}_2} + \cancel{\hat{a}_2^\dagger\hat{a}_1^\dagger}$$

Pauli operators
in the qubit subspace

C4 code stabilizer:

$$\hat{S}_{C4} = -\hat{c}_1\hat{c}_2\hat{c}_3\hat{c}_4$$

More Pauli operators:

$$\hat{S}_{C4}\hat{Z} = i\hat{c}_1\hat{c}_4$$

$$\hat{S}_{C4}\hat{X} = i\hat{c}_3\hat{c}_4$$

Fermionic Gaussian states and Fermionic Linear Optics

Definition: A pure fermionic state is Gaussian iff it is a ground state of a quadratic Hamiltonian. $\hat{H} = i \sum_{j,k}^{2N} A_{jk} \hat{c}_j \hat{c}_k$

Definition: FLO gates take Gaussian states to Gaussian states.

Gaussian states[1,2]:

1. Take a general quadratic Hamiltonian
2. Determine its ground state
3. Show that this state can be fully characterized by the covariance matrix
4. Show that every expectation value can be calculated efficiently from the covariance matrix

FLO Gates[1,2]:

1. Define the two types of FLO gates: Non-interacting time evolution, measurement of Majorana pairs
2. Show the covariance matrix transformations representing FLO gates.

A concrete example

[1] Bravyi, Sergey and Robert König. “Classical simulation of dissipative fermionic linear optics.” Quantum Inf. Comput. 12 (2011)
[2] Terhal, Barbara M., and David P. DiVincenzo. "Classical simulation of noninteracting-fermion quantum circuits." *Physical Review A* 65.3 (2002)

Pure Gaussian states

Take a system with N fermionic modes (N/2 qubits 2N Majoranas)

General quadratic Hamiltonian:

$$\hat{H} = i\sum_{j,k}^{2N} A_{jk}\hat{c}_j\hat{c}_k$$ (+const) 0 energy offset: $A_{jj} = 0$

$$\hat{H}^\dagger = \hat{H} \longrightarrow A^*_{jk} = A_{jk}, \quad A_{kj} = -A_{jk}$$

$\underline{\underline{A}}$ is a real, antisymmetric matrix

<u>Theorem</u>: $\exists!$ $\underline{\underline{R}} \in SO(2N)$ real, orthogonal matrix: $\underline{\underline{R}}\ \underline{\underline{A}}\ \underline{\underline{R}}^T = \bigoplus_{n=1}^{N} \begin{bmatrix} 0 & \alpha_n \\ -\alpha_n & 0 \end{bmatrix}$, $\alpha_n \in \mathbb{R}$

Introduce new Majorana fermions:

$$\hat{c}'_j = \sum_k R_{jk}\hat{c}_k$$

$$\hat{H} = i\sum_{j,k,l,m,n,o} \hat{c}_j R_{jk} R_{kl} A_{lm} R_{nm} R_{no}\hat{c}_o = 2i\sum_n^N \alpha_n \hat{c}'_{2n-1}\hat{c}'_{2n}$$

Ground state:

$$-\frac{\alpha_n}{|\alpha_n|} i\hat{c}'_{2n-1}\hat{c}'_{2n}|GS\rangle = |GS\rangle \quad \forall n$$

$$|GS\rangle\langle GS| = \frac{1}{2^N}\prod_n^N \left(1 + \frac{\alpha_n}{|\alpha_n|} i\hat{c}'_{2n-1}\hat{c}'_{2n}\right)$$

Covariance matrix

Covariance matrix:

$$M_{jk}(|\psi\rangle) = \langle\psi|i\hat{c}_j\hat{c}_k|\psi\rangle - i\delta_{jk}$$

If $|\psi\rangle$ is Gaussian $\longrightarrow$ $\underline{\underline{R}}\ \underline{\underline{M}}\ \underline{\underline{R}}^T = \bigoplus_{n=1}^{N} \begin{bmatrix} 0 & -\alpha_n/|\alpha_n| \\ \alpha_n/|\alpha_n| & 0 \end{bmatrix}$

Wick's theorem:

$$i^n\langle GS|\hat{c}_{j_1}\hat{c}_{j_2}...\hat{c}_{j_{2n}}|GS\rangle = \mathrm{Pf}(M(|GS\rangle)_{j_1,j_2,...,j_{2n}})$$

Expectation values can be expressed with the covariance matrix!

Example:

$$-\langle GS|\hat{c}_j\hat{c}_k\hat{c}_l\hat{c}_m|GS\rangle = M_{jk}M_{lm} - M_{jl}M_{km} + M_{jm}M_{kl}$$

FLO gates

Definition: FLO gates take Gaussian states to Gaussian states.

1. Non-interacting time evolution

$\exp(-t\hat{c}_j\hat{c}_k)$ (Generally: $\exp(-t\sum_{jk} H_{jk}\hat{c}_j\hat{c}_k)$)

$$|GS\rangle \longleftrightarrow \hat{H}$$

$$\exp(-t\hat{c}_j\hat{c}_k)|GS\rangle \longleftrightarrow \exp(-t\hat{c}_j\hat{c}_k)\hat{H}\exp(t\hat{c}_j\hat{c}_k)$$

2. Measurement of Majorana pairs

$$\left.\begin{array}{l} \dfrac{1}{\sqrt{p}}\dfrac{1+i\hat{c}_j\hat{c}_k}{2}|GS\rangle \\[2ex] p = \dfrac{1+M_{jk}}{2} \end{array}\right\}$$

Calculation of probabilities
+
Application of projectors

Covariance matrix transformations:

$$M'_{pq} = \langle GS|\exp(t\hat{c}_j\hat{c}_k)i\hat{c}_p\hat{c}_q\exp(-t\hat{c}_j\hat{c}_k)|GS\rangle - i\delta_{pq}$$

$$M'_{pq} = \frac{\langle GS|(1+i\hat{c}_j\hat{c}_k)i\hat{c}_p\hat{c}_q(1+i\hat{c}_j\hat{c}_k)|GS\rangle}{2\langle GS|(1+i\hat{c}_j\hat{c}_k)|GS\rangle} - i\delta_{jk}$$

$\xrightarrow[\text{theorem}]{\text{Wick's}}$ $\underline{\underline{M'}}$ can be constructed from $\underline{\underline{M}}$

Algorithms for covariance matrix transformations

1. Non-interacting time evolution

$$\exp(-\theta \hat{c}_p \hat{c}_q)$$

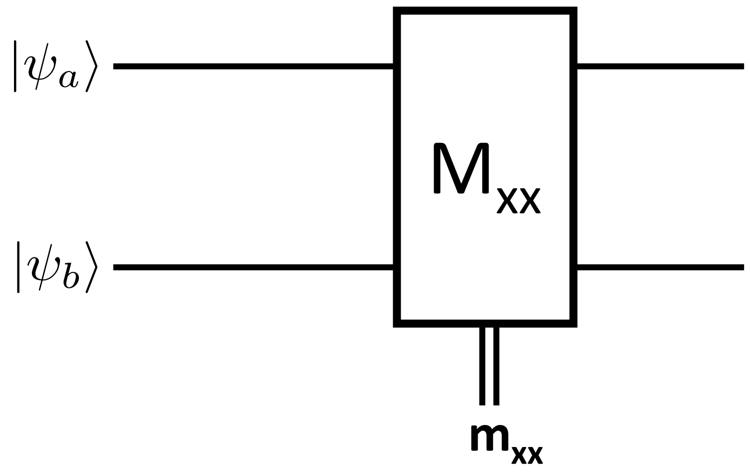
```
Algorithm 1: Rotation (M,θ,p,q)
    $M'[p,:] \leftarrow M[p,:]\cos(2\theta) - M[q,:]\sin(2\theta)$
    $M'[q,:] \leftarrow M[q,:]\cos(2\theta) + M[p,:]\sin(2\theta)$
    $M'[:,p] \leftarrow M[:,p]\cos(2\theta) - M[:,q]\sin(2\theta)$
    $M'[:,q] \leftarrow M[:,q]\cos(2\theta) + M[:,p]\sin(2\theta)$
    $M'[p,p] \leftarrow 0$
    $M'[q,q] \leftarrow 0$
    $M'[p,q] \leftarrow$
      $M[p,q]\cos^2(2\theta) - M[q,p]\sin^2(2\theta)$
    $M'[q,p] \leftarrow$
      $-M[p,q]\cos^2(2\theta) + M[q,p]\sin^2(2\theta)$
    return $M'$
```

2. Measurement of Majorana pairs

$$\frac{1}{\sqrt{p}} \frac{1 + i\hat{c}_p\hat{c}_q}{2} |GS\rangle$$

```
Algorithm 2: Measurement (M,p,q)
    probability $\leftarrow (1/2)(1 + M[p,q])$
    if $p \neq 0$ then
        $\mathbf{K} \leftarrow M[p,:]$
        $\mathbf{L} \leftarrow M[q,:]$
        $M' \leftarrow M + (1/2p)(\mathbf{L}\mathbf{K}^T - \mathbf{K}\mathbf{L}^T)$
        $M'[p,:] \leftarrow 0$
        $M'[q,:] \leftarrow 0$
        $M'[:,p] \leftarrow 0$
        $M'[:,q] \leftarrow 0$
        $M'[p,q] \leftarrow 1$
        $M'[q,p] \leftarrow -1$
    end
    return $(M'$, probability$)$
```

XX measurement can be simulated with FLO gates



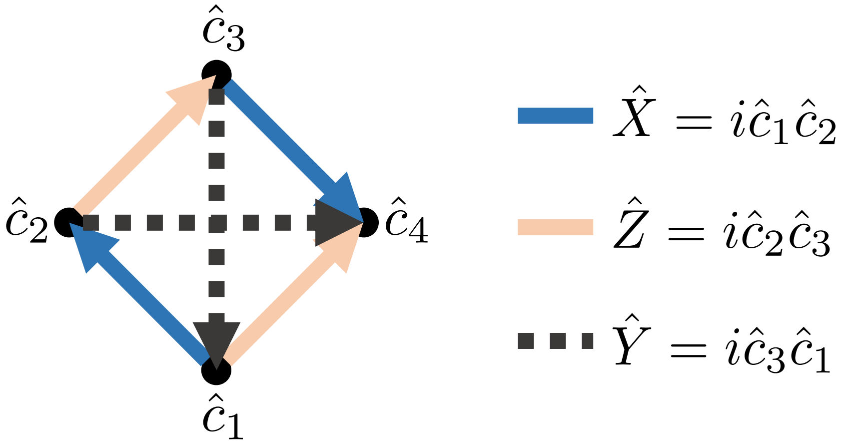
$\mathbf{m_{xx}}$ can be sampled efficiently with Fermionic Linear Optics

$|\psi_a\rangle \otimes |\psi_b\rangle$ is a Gaussian state

XX measurement is not an FLO gate

→ How to sample the measurement outcome?

Single qubit states are Gaussian



$$\hat{X} = i\hat{c}_1\hat{c}_2$$

$$\hat{Z} = i\hat{c}_2\hat{c}_3$$

$$\hat{Y} = i\hat{c}_3\hat{c}_1$$

C4 stabilizer: $\hat{S}_{C4} = -\hat{c}_1\hat{c}_2\hat{c}_3\hat{c}_4$

Single qubit states:

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$$
$$= \exp\left(-i\frac{\phi-\pi/2}{2}\hat{Z}\right)\exp\left(i\frac{\theta}{2}\hat{X}\right)|0\rangle$$
$$= \exp\left(\frac{\phi-\pi/2}{2}\hat{c}_2\hat{c}_3\right)\exp\left(-\frac{\theta}{2}\hat{c}_1\hat{c}_2\right)|0\rangle$$

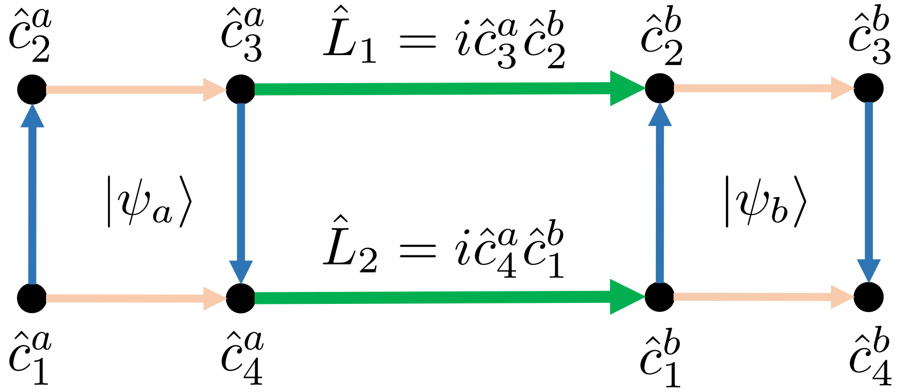
FLO

Gaussian

The covariance matrix:

$$\underline{\underline{M}} = \begin{bmatrix} 0 & \sin(\theta)\cos(\phi) & -\sin(\theta)\sin(\phi) & \cos(\theta) \\ -\sin(\theta)\cos(\phi) & 0 & \cos(\theta) & \sin(\theta)\sin(\phi) \\ \sin(\theta)\sin(\phi) & -\cos(\theta) & 0 & \sin(\theta)\cos(\phi) \\ -\cos(\theta) & -\sin(\theta)\sin(\phi) & -\sin(\theta)\cos(\phi) & 0 \end{bmatrix}$$

Sampling the XX measurement



$$\hat{L}_1\hat{L}_2 = \hat{X}^a\hat{X}^b$$

Sampling the measurement outcome of the XX measurement can be done by measuring **link-operators**.

Measurement of link-operators:

$$\hat{L}_1 \to l_1 \qquad \hat{L}_2 \to l_2$$

$$P(m_{xx}=1) = P(l_1 = 1 \wedge l_2 = 1) + P(l_1 = -1 \wedge l_2 = -1)$$

$$P(m_{xx}=-1) = P(l_1 = 1 \wedge l_2 = -1) + P(l_1 = -1 \wedge l_2 = 1)$$

$$|\psi_a\rangle = \cos(\theta_a/2)|0\rangle + e^{i\phi_a}\sin(\theta_a/2)|1\rangle$$

$$|\psi_b\rangle = \cos(\theta_b/2)|0\rangle + e^{i\phi_b}\sin(\theta_b/2)|1\rangle$$

$$\longrightarrow \quad P(m_{xx}=1) = \frac{1}{2}(1+\sin(\theta_a)\sin(\theta_b)\cos(\phi_a)\cos(\phi_b))$$

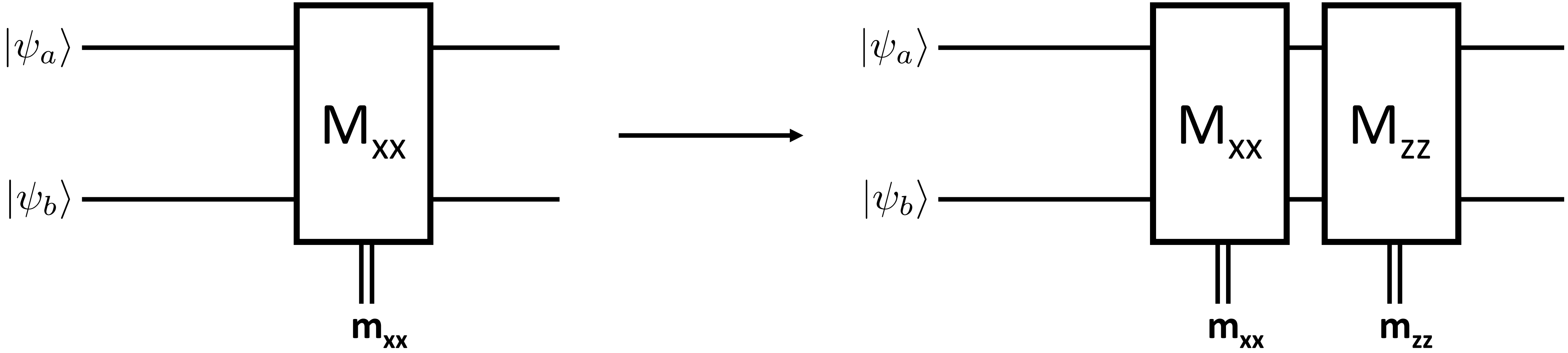
Fermionic Linear Optics is useless in this case!

XX measurement can be sampled with individual X measurements

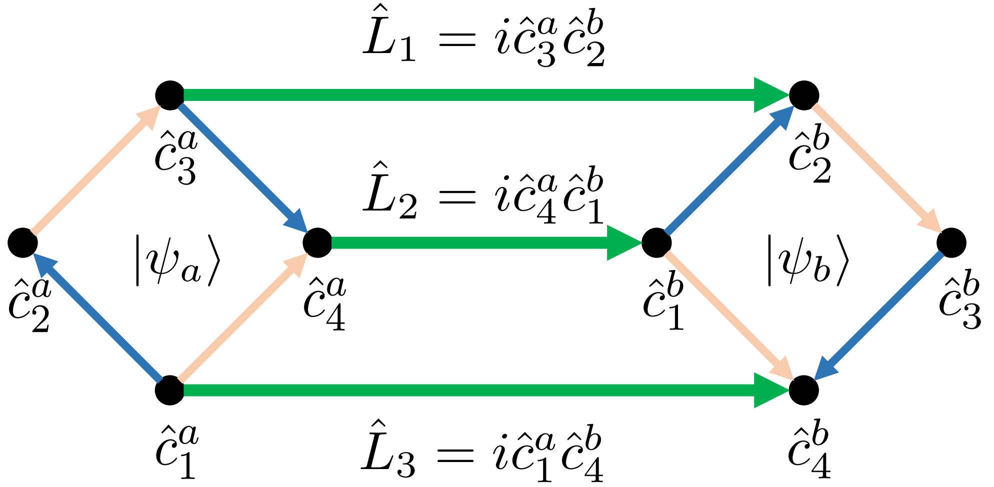
$$P(m_{xx}=1) = P(m_{x_1} = 1 \wedge m_{x_2} = 1) + P(m_{x_1} = -1 \wedge m_{x_2} = -1)$$

$$P(m_{xx}=-1) = P(m_{x_1} = 1 \wedge m_{x_2} = -1) + P(m_{x_1} = -1 \wedge m_{x_2} = 1)$$

Sampling XX and ZZ measurements



m$_{xx}$ and **m**$_{zz}$ can be sampled efficiently with FLO, but NOT with single qubit measurements.



$$\hat{L}_1\hat{L}_2 = \hat{X}^a\hat{X}^b, \qquad \hat{L}_2\hat{L}_3 = \hat{Z}^a\hat{Z}^b$$

Measurement of 3 **link-operators** is enough:

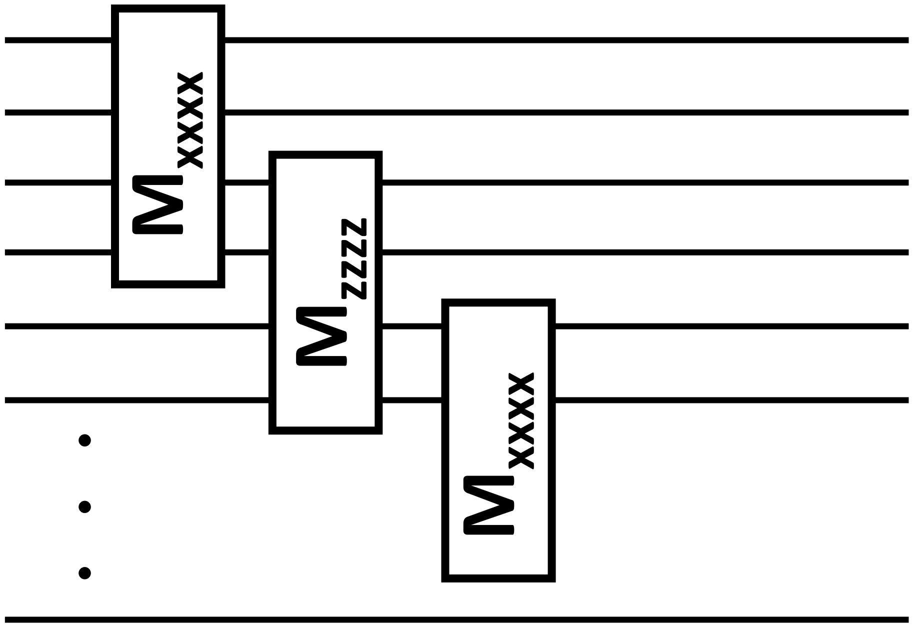
$$P(m_{xx} = 1) = P(l_1 = 1 \wedge l_2 = 1) + P(l_1 = -1 \wedge l_2 = -1)$$

$$P(m_{xx} = -1) = P(l_1 = 1 \wedge l_2 = -1) + P(l_1 = -1 \wedge l_2 = 1)$$

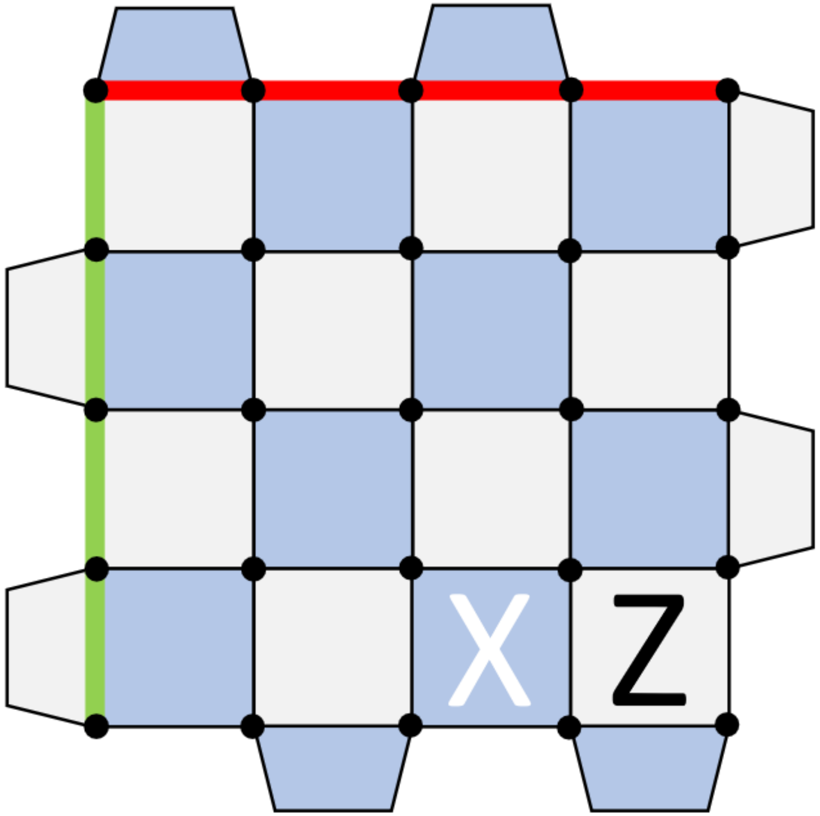
$$P(m_{zz} = 1) = P(l_2 = 1 \wedge l_3 = 1) + P(l_2 = -1 \wedge l_3 = -1)$$

$$P(m_{zz} = -1) = P(l_2 = 1 \wedge l_3 = -1) + P(l_2 = -1 \wedge l_3 = 1)$$

Simulation of quantum error correction



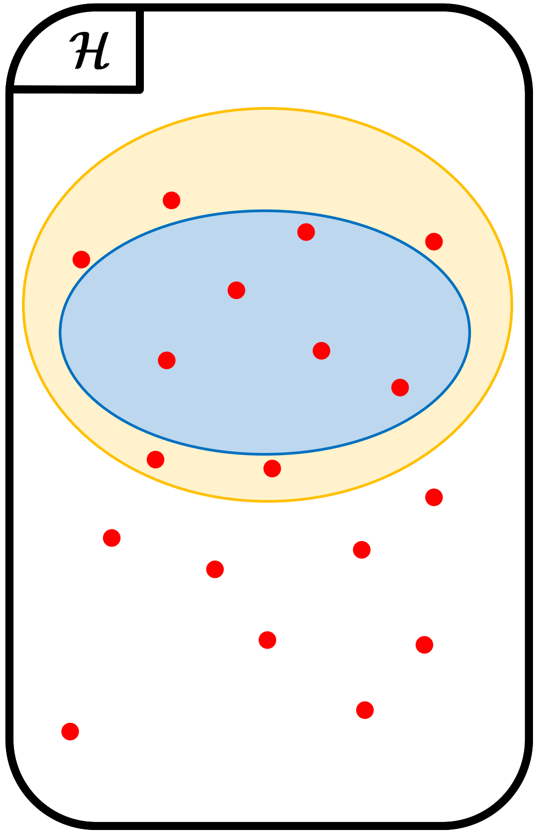
Surface code layout



- Noisy state preparation[1]
- Coherent Z errors[1]
- Coherent Z errors in planar graphs[2]
- Coherent Z errors and readout errors[3]

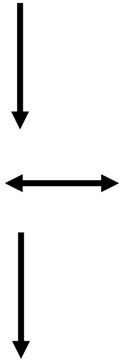
[1] Bravyi, Sergey, et al. "Correcting coherent errors with surface codes." *npj Quantum Information* 4.1 (2018)
[2] Venn, F., and B. Béri. "Error-correction and noise-decoherence thresholds for coherent errors in planar-graph surface codes. " *Physical Review Research* 2.4 (2020)
[3] Márton, Áron and János K. Asbóth. “Coherent errors and readout errors in the surface code.” *Quantum* (2023)

Summary



Classical simulation of quantum circuits:

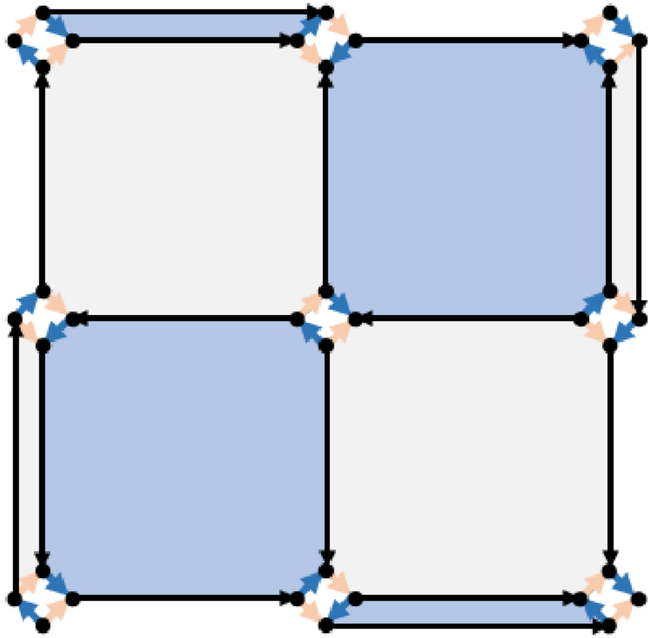
1. Product states
2. Clifford simulations
3. **Fermionic Linear Optics**



Qubit Majorana mapping (C4 code)

Pure Gaussian states ⟷ Covariance matrix

FLO gates: - Free time evolution
- Measurement of Majorana pairs



Possible application:
Quantum error correction

Thank you for your attention!