Quantum Walks

András Gilyén

ELTE kvantum-infó szeminárium 2020 szeptember 24

Outline

- Coined quantum walks
- Szegedy quantum walks
- Continuous-time quantum walks

Coined quantum walks

(Classical) Random walk on the line

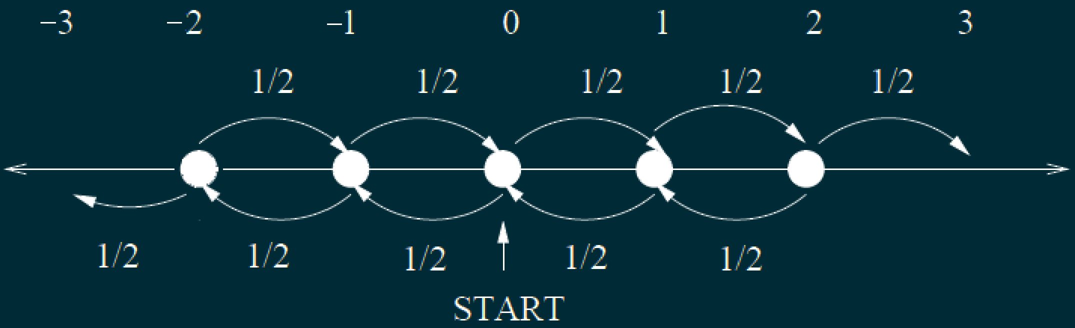
In each step we toss a coin:

- if **heads** we go one step **left**
- if **tails** we go one step **right**

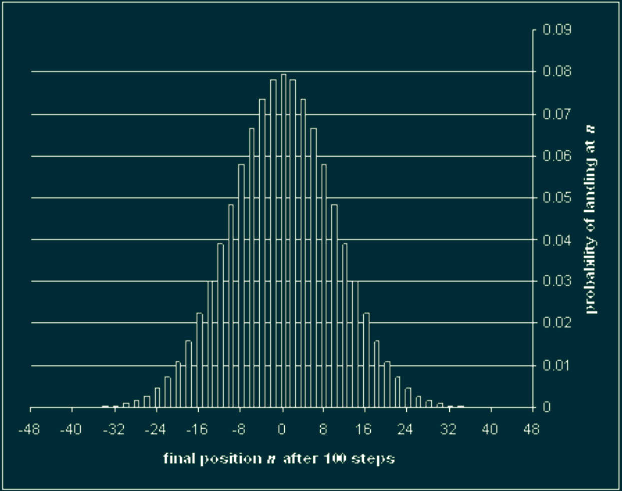
(Classical) Random walk on the line

In each step we toss a coin:

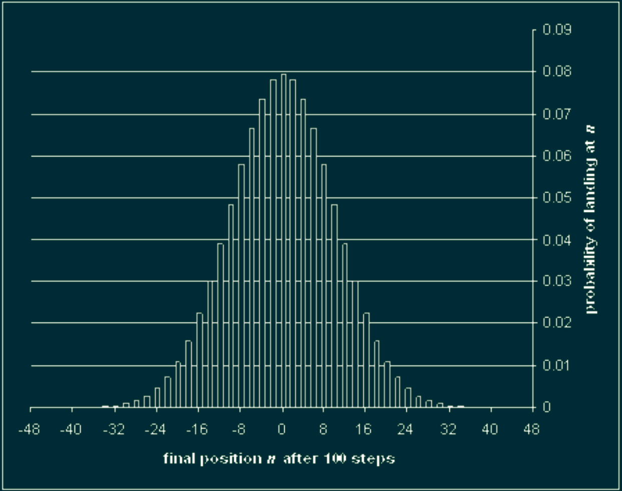
- if **heads** we go one step **left**
- if **tails** we go one step **right**



The distribution after T steps is roughly $N(0, \sqrt{T})$

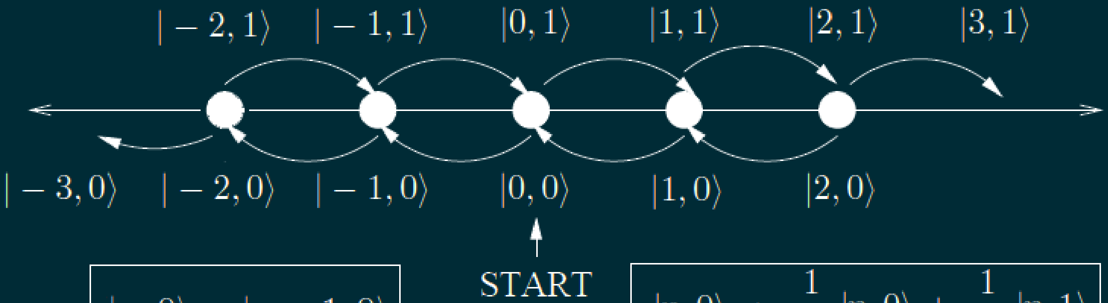


The distribution after T steps is roughly $N(0, \sqrt{T})$



- The probability of being further than $\sim \sqrt{T}$ is small.
- Being at some specific point of the interval $[-\sqrt{T}, \sqrt{T}]$ has chance $\sim 1/\sqrt{T}$, i.e., the distribution is roughly uniform on this interval.

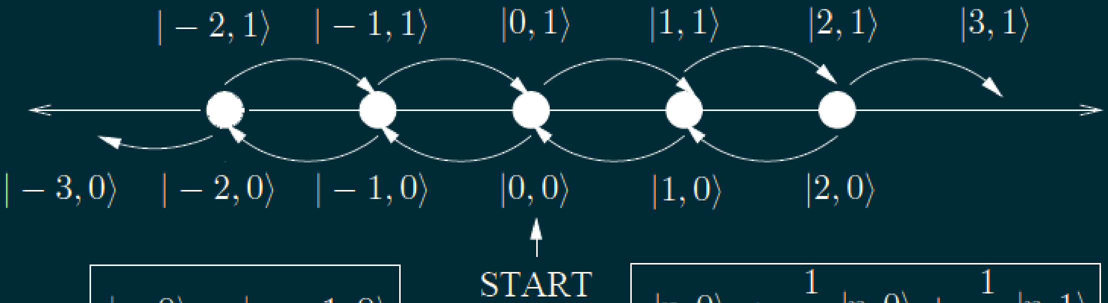
Quantum walk on the line



$$S = \boxed{\begin{array}{l} |n,0\rangle \rightarrow |n-1,0\rangle \\ \\ |n,1\rangle \rightarrow |n+1,1\rangle \end{array}}$$

$$C = \boxed{\begin{array}{l} |n,0\rangle \rightarrow \frac{1}{\sqrt{2}}|n,0\rangle + \frac{1}{\sqrt{2}}|n,1\rangle \\ |n,1\rangle \rightarrow \frac{1}{\sqrt{2}}|n,0\rangle - \frac{1}{\sqrt{2}}|n,1\rangle \end{array}}$$

Quantum walk on the line



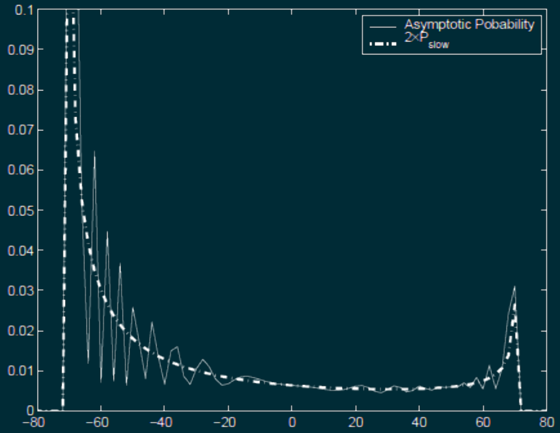
$$S = \begin{array}{|l|} \hline |n,0\rangle \rightarrow |n-1,0\rangle \\ |n,1\rangle \rightarrow |n+1,1\rangle \\ \hline \end{array}$$

$$C = \begin{array}{|l|} \hline |n,0\rangle \rightarrow \frac{1}{\sqrt{2}}|n,0\rangle + \frac{1}{\sqrt{2}}|n,1\rangle \\ |n,1\rangle \rightarrow \frac{1}{\sqrt{2}}|n,0\rangle - \frac{1}{\sqrt{2}}|n,1\rangle \\ \hline \end{array}$$

One step of the walk: *SC* i.e., first apply *C* and then apply *S*

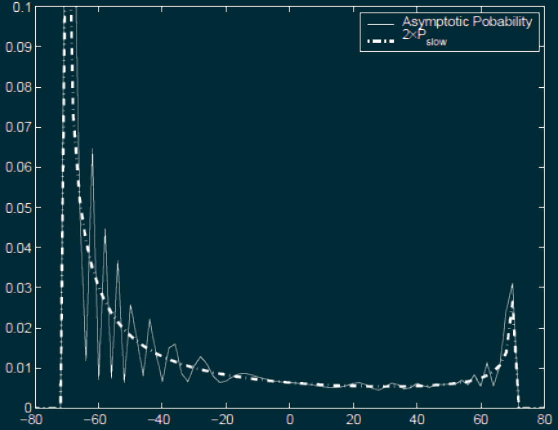
- Quantum walk is deterministic and reversible!
- Only the measurement introduces probability.

Distribution after T steps – upon measurement



The probability distribution of the quantum random walk with Hadamard coin starting in $|0, 0\rangle$ after $T = 100$ steps.

Distribution after T steps – upon measurement



The probability distribution of the quantum random walk with Hadamard coin starting in $|0,0\rangle$ after $T = 100$ steps.

Main differences

- Peak around $T/\sqrt{2} \rightarrow$ ballistic spreading
- Between $-T/2$ and $T/2$ the distribution is $\sim$ uniform $\rightarrow$ quadratically faster mixing

Szegedy quantum walks

Discrete-time quantum / random walks

Discrete-time Markov-chain on a weighted graph

Transition probability in one step (stochastic matrix)

$$P_{vu} = \Pr(\text{step to } v \,|\, \text{being at } u) = \frac{w_{vu}}{\sum_{v' \in U} w_{v'u}}$$

Discrete-time quantum / random walks

Discrete-time Markov-chain on a weighted graph

Transition probability in one step (stochastic matrix)

$$P_{vu} = \Pr(\text{step to } v \,|\, \text{being at } u) = \frac{w_{vu}}{\sum_{v' \in U} w_{v'u}}$$

A unitary implementing the update

$$U \colon |0\rangle|u\rangle \mapsto \sum_{v \in V} \sqrt{P_{vu}}|v\rangle|u\rangle$$

Discrete-time quantum / random walks

Discrete-time Markov-chain on a weighted graph

Transition probability in one step (stochastic matrix)

$$P_{vu} = \Pr(\text{step to } v \,|\, \text{being at } u) = \frac{w_{vu}}{\sum_{v' \in U} w_{v'u}}$$

A unitary implementing the update

$$U\colon |0\rangle|u\rangle \mapsto \sum_{v \in V} \sqrt{P_{vu}}|v\rangle|u\rangle$$

Generic coin operator

$$S := \mathrm{SWAP}$$
$$C := U((2|0\rangle\langle 0| \otimes I) - I)U^{\dagger}$$

Discrete-time quantum / random walks

Discrete-time Markov-chain on a weighted graph

Transition probability in one step (stochastic matrix)

$$P_{vu} = \Pr(\text{step to } v \,|\, \text{being at } u) = \frac{w_{vu}}{\sum_{v' \in U} w_{v'u}}$$

A unitary implementing the update

$$U \colon |0\rangle|u\rangle \mapsto \sum_{v \in V} \sqrt{P_{vu}}|v\rangle|u\rangle$$

How to erase history? The Szegedy quantum walk operator:

$$W' := U^\dagger \cdot \mathrm{SWAP} \cdot U$$
$$W := U^\dagger \cdot \mathrm{SWAP} \cdot U((2|0\rangle\langle 0| \otimes I) - I)$$

Understanding Szegedy's quantum walk operator

For simplicity let us assume $P_{uv} = P_{vu}$, i.e., the total weight of vertices is constants.

Understanding Szegedy's quantum walk operator

For simplicity let us assume $P_{uv} = P_{vu}$, i.e., the total weight of vertices is constants.

A block-encoding of the Markov chain: $(\langle 0| \otimes I)W'(|0\rangle \otimes I) = P$

Understanding Szegedy's quantum walk operator

For simplicity let us assume $P_{uv} = P_{vu}$, i.e., the total weight of vertices is constants.

A block-encoding of the Markov chain: $(\langle 0| \otimes I)W'(|0\rangle \otimes I) = P$

Proof:

$$\langle 0|\langle u|W'|0\rangle|v\rangle = \langle 0|\langle u|U^\dagger \cdot \mathrm{SWAP} \cdot U|0\rangle|v\rangle = \left(\sum_{v' \in V} \sqrt{P_{v'u}}|v'\rangle|u\rangle\right)^\dagger \mathrm{SWAP} \left(\sum_{u' \in V} \sqrt{P_{u'v}}|u'\rangle|v\rangle\right)$$

Understanding Szegedy's quantum walk operator

For simplicity let us assume $P_{uv} = P_{vu}$, i.e., the total weight of vertices is constants.

A block-encoding of the Markov chain: $(\langle 0| \otimes I) W' (|0\rangle \otimes I) = P$

Proof:

$$\langle 0|\langle u|W'|0\rangle|v\rangle = \langle 0|\langle u|U^\dagger \cdot \mathrm{SWAP} \cdot U|0\rangle|v\rangle = \left(\sum_{v' \in V} \sqrt{P_{v'u}}|v'\rangle|u\rangle\right)^\dagger \mathrm{SWAP} \left(\sum_{u' \in V} \sqrt{P_{u'v}}|u'\rangle|v\rangle\right)$$

Multiple steps of the quantum walk: $(\langle 0| \otimes I) W^k (|0\rangle \otimes I) = T_k(P)$

[$T_k(x) = \cos(k \arccos(x))$ Chebyshev polynomials: $T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)$]

Understanding Szegedy's quantum walk operator

For simplicity let us assume $P_{uv} = P_{vu}$, i.e., the total weight of vertices is constants.

A block-encoding of the Markov chain: $(\langle 0| \otimes I)W'(|0\rangle \otimes I) = P$

Proof:

$$\langle 0|\langle u|W'|0\rangle|v\rangle = \langle 0|\langle u|U^\dagger \cdot \mathrm{SWAP} \cdot U|0\rangle|v\rangle = \left(\sum_{v'\in V} \sqrt{P_{v'u}}|v'\rangle|u\rangle\right)^\dagger \mathrm{SWAP}\left(\sum_{u'\in V} \sqrt{P_{u'v}}|u'\rangle|v\rangle\right)$$

Multiple steps of the quantum walk: $(\langle 0| \otimes I)W^k(|0\rangle \otimes I) = T_k(P)$

[$T_k(x) = \cos(k \arccos(x))$ Chebyshev polynomials: $T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)$]

Proof: Proceed by induction, observe $T_0(P) = I\checkmark$, $T_1(P) = P\checkmark$

Understanding Szegedy's quantum walk operator

For simplicity let us assume $P_{uv} = P_{vu}$, i.e., the total weight of vertices is constants.

A block-encoding of the Markov chain: $(\langle 0| \otimes I)W'(|0\rangle \otimes I) = P$

Proof:

$$\langle 0|\langle u|W'|0\rangle|v\rangle = \langle 0|\langle u|U^\dagger \cdot \mathrm{SWAP} \cdot U|0\rangle|v\rangle = \left(\sum_{v'\in V} \sqrt{P_{v'u}}|v'\rangle|u\rangle\right)^\dagger \mathrm{SWAP}\left(\sum_{u'\in V} \sqrt{P_{u'v}}|u'\rangle|v\rangle\right)$$

Multiple steps of the quantum walk: $(\langle 0| \otimes I)W^k(|0\rangle \otimes I) = T_k(P)$

[$T_k(x) = \cos(k \arccos(x))$ Chebyshev polynomials: $T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)$]

Proof: Proceed by induction, observe $T_0(P) = I\,\checkmark$, $T_1(P) = P\,\checkmark$

$$(\langle 0| \otimes I)W^{k+1}(|0\rangle \otimes I) = (\langle 0| \otimes I)W'((2|0\rangle\langle 0| \otimes I) - I)W^k(|0\rangle \otimes I) =$$

$$= \underbrace{(\langle 0| \otimes I)W'(2|0\rangle}_{2P}\underbrace{\langle 0| \otimes I)W^k(|0\rangle \otimes I)}_{T_k(P)} - \underbrace{(\langle 0| \otimes I)W^{k-1}(|0\rangle \otimes I)}_{T_{k-1}(P)}$$

Understanding Szegedy's quantum walk operator

For simplicity let us assume $P_{uv} = P_{vu}$, i.e., the total weight of vertices is constants.

A block-encoding of the Markov chain: $(\langle 0| \otimes I) W' (|0\rangle \otimes I) = P$

Proof:

$$\langle 0|\langle u|W'|0\rangle|v\rangle = \langle 0|\langle u|U^\dagger \cdot \mathrm{SWAP} \cdot U|0\rangle|v\rangle = \left(\sum_{v'\in V} \sqrt{P_{v'u}}|v'\rangle|u\rangle\right)^\dagger \mathrm{SWAP} \left(\sum_{u'\in V} \sqrt{P_{u'v}}|u'\rangle|v\rangle\right)$$

Multiple steps of the quantum walk: $(\langle 0| \otimes I) W^k (|0\rangle \otimes I) = T_k(P)$

$[T_k(x) = \cos(k \arccos(x))$ Chebyshev polynomials: $T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)]$

Proof: Proceed by induction, observe $T_0(P) = I \checkmark$, $T_1(P) = P \checkmark$

$$\begin{aligned}(\langle 0| \otimes I) W^{k+1} (|0\rangle \otimes I) &= (\langle 0| \otimes I) W' ((2|0\rangle\langle 0| \otimes I) - I) W^k (|0\rangle \otimes I) = \\ &= \underbrace{(\langle 0| \otimes I) W' (2|0\rangle}_{2P} \underbrace{\langle 0| \otimes I) W^k (|0\rangle \otimes I)}_{T_k(P)} - \underbrace{(\langle 0| \otimes I) W^{k-1} (|0\rangle \otimes I)}_{T_{k-1}(P)}\end{aligned}$$

Are we happy with Chebyshev polynomials?

Linear combination of (non-)unitary mat. [Childs & Wiebe ’12, Berry et al. ’15]

Suppose that $U = \sum_i |i\rangle\langle i| \otimes U_i$, and $Q : |0\rangle \mapsto \sum_i \sqrt{q_i}|i\rangle$ for $q_i \in [0, 1]$.

Are we happy with Chebyshev polynomials?

Linear combination of (non-)unitary mat. [Childs & Wiebe '12, Berry et al. '15]

Suppose that $U = \sum_i |i\rangle\langle i| \otimes U_i$, and $Q : |0\rangle \mapsto \sum_i \sqrt{q_i}|i\rangle$ for $q_i \in [0,1]$.

Then the top-left corner $(\langle 0| \otimes I)(Q^\dagger \otimes I)U(Q \otimes I)(|0\rangle \otimes I)$ is $\sum_i q_i U_i$.

Are we happy with Chebyshev polynomials?

Linear combination of (non-)unitary mat. [Childs & Wiebe '12, Berry et al. '15]

Suppose that $U = \sum_i |i\rangle\langle i| \otimes U_i$, and $Q : |0\rangle \mapsto \sum_i \sqrt{q_i}|i\rangle$ for $q_i \in [0, 1]$.

Then the top-left corner $(\langle 0| \otimes I)(Q^\dagger \otimes I)U(Q \otimes I)(|0\rangle \otimes I)$ is $\sum_i q_i U_i$.

In particular if $(\langle 0| \otimes I)U_k(|0\rangle \otimes I) = T_k(P)$, then

$$(\langle 0|\langle 0| \otimes I)(Q^\dagger \otimes I)U(Q \otimes I)(|0\rangle|0\rangle \otimes I) = \sum_i q_k T_k(P).$$

Are we happy with Chebyshev polynomials?

Linear combination of (non-)unitary mat. [Childs & Wiebe '12, Berry et al. '15]

Suppose that $U = \sum_i |i\rangle\langle i| \otimes U_i$, and $Q : |0\rangle \mapsto \sum_i \sqrt{q_i}|i\rangle$ for $q_i \in [0, 1]$.

Then the top-left corner $(\langle 0| \otimes I)(Q^\dagger \otimes I)U(Q \otimes I)(|0\rangle \otimes I)$ is $\sum_i q_i U_i$.

In particular if $(\langle 0| \otimes I)U_k(|0\rangle \otimes I) = T_k(P)$, then

$$(\langle 0|\langle 0| \otimes I)(Q^\dagger \otimes I)U(Q \otimes I)(|0\rangle|0\rangle \otimes I) = \sum_i q_k T_k(P).$$

Used for Hamiltonian simulation, and much more!

Are we happy with Chebyshev polynomials?

Linear combination of (non-)unitary mat. [Childs & Wiebe '12, Berry et al. '15]

Suppose that $U = \sum_i |i\rangle\langle i| \otimes U_i$, and $Q : |0\rangle \mapsto \sum_i \sqrt{q_i}|i\rangle$ for $q_i \in [0,1]$.
Then the top-left corner $(\langle 0| \otimes I)(Q^\dagger \otimes I)U(Q \otimes I)(|0\rangle \otimes I)$ is $\sum_i q_i U_i$.
In particular if $(\langle 0| \otimes I)U_k(|0\rangle \otimes I) = T_k(P)$, then

$$(\langle 0|\langle 0| \otimes I)(Q^\dagger \otimes I)U(Q \otimes I)(|0\rangle|0\rangle \otimes I) = \sum_i q_k T_k(P).$$

Used for Hamiltonian simulation, and much more!

Corollary: Quantum fast-forwarding (Apers & Sarlette 2018)

We can implement a unitary V such that

$$(\langle 0| \otimes I)V(|0\rangle \otimes I) \overset{\varepsilon}{\approx} P^t$$

with using only $O\left(\sqrt{t \log(1/\varepsilon)}\right)$ quantum walk steps.

Are we happy with Chebyshev polynomials?

Linear combination of (non-)unitary mat. [Childs & Wiebe '12, Berry et al. '15]

Suppose that $U = \sum_i |i\rangle\langle i| \otimes U_i$, and $Q : |0\rangle \mapsto \sum_i \sqrt{q_i}|i\rangle$ for $q_i \in [0,1]$.
Then the top-left corner $(\langle 0| \otimes I)(Q^\dagger \otimes I)U(Q \otimes I)(|0\rangle \otimes I)$ is $\sum_i q_i U_i$.
In particular if $(\langle 0| \otimes I)U_k(|0\rangle \otimes I) = T_k(P)$, then

$$(\langle 0|\langle 0| \otimes I)(Q^\dagger \otimes I)U(Q \otimes I)(|0\rangle|0\rangle \otimes I) = \sum_i q_k T_k(P).$$

Used for Hamiltonian simulation, and much more!

Corollary: Quantum fast-forwarding (Apers & Sarlette 2018)

We can implement a unitary V such that

$$(\langle 0| \otimes I)V(|0\rangle \otimes I) \overset{\varepsilon}{\approx} P^t$$

with using only $\mathcal{O}\left(\sqrt{t \log(1/\varepsilon)}\right)$ quantum walk steps. (Proof: $x^t \approx \sum_{k=0}^{\approx\sqrt{t}} T_k(x)$)

Szegedy quantum walk based search

Suppose we have some unknown marked vertices $M \subset V$.

Quadratically faster hitting

Hitting time: expected time to hit a marked vertex starting from the stationary distr.
Starting from the quantum state $\sum_{v\in V}\sqrt{\pi_v}|v\rangle$ we can

- detect the presence of marked vertices ($M \neq 0$) in time $\mathcal{O}\big(\sqrt{HT}\big)$ (Szegedy 2004)
- find a marked vertex in time $\mathcal{O}\big(\frac{1}{\sqrt{\delta\varepsilon}}\big)$ (Magniez, Nayak, Roland, Sántha 2006)
- find a marked vertex in time $\widetilde{\mathcal{O}}\big(\sqrt{HT}\big)$ (Ambainis, G, Jeffery, Kokainis 2019)

Szegedy quantum walk based search

Suppose we have some unknown marked vertices $M \subset V$.

Quadratically faster hitting

Hitting time: expected time to hit a marked vertex starting from the stationary distr.
Starting from the quantum state $\sum_{v \in V} \sqrt{\pi_v}|v\rangle$ we can

- detect the presence of marked vertices ($M \neq 0$) in time $\mathcal{O}\left(\sqrt{HT}\right)$ (Szegedy 2004)
- find a marked vertex in time $\mathcal{O}\left(\frac{1}{\sqrt{\delta\varepsilon}}\right)$ (Magniez, Nayak, Roland, Sántha 2006)
- find a marked vertex in time $\widetilde{\mathcal{O}}\left(\sqrt{HT}\right)$ (Ambainis, G, Jeffery, Kokainis 2019)

Starting from arbitrary distributions

Starting from distribution σ on some vertices we can

- detect marked vertices in square-root commute time $\mathcal{O}\left(\sqrt{C_{\sigma,M}}\right)$ (Belovs 2013)
- find a marked vertex in time $\widetilde{\mathcal{O}}\left(\sqrt{C_{\sigma,M}}\right)$ (Piddock; Apers, G, Jeffery 2019)

Walks on the Johnson graph (Sántha `arXiv:0808.0059`)

Vertices: $\{S \subset N \colon |S| = K\}$; Edges: $\{(S, S') \colon |S \triangle S'| = 2\}$

Walks on the Johnson graph (Sántha arXiv:0808.0059)

Vertices: $\{S \subset N \colon |S| = K\}$; Edges: $\{(S, S') \colon |S \triangle S'| = 2\}$

Element Distinctness

- Black box: Computes f on inputs corresponding to elements of $[n]$
- Question: Are there any $i \neq j \in [n] \times [n]$ such that $f(i) = f(j)$?
- Query complexity: $\mathcal{O}(n^{2/3})$ (Ambainis 2003) $\Omega(n^{2/3})$ (Aaronson & Shi 2001)

Walks on the Johnson graph (Sántha `arXiv:0808.0059`)

Vertices: $\{S \subset N : |S| = K\}$; Edges: $\{(S, S') : |S \triangle S'| = 2\}$

Element Distinctness

- Black box: Computes f on inputs corresponding to elements of $[n]$
- Question: Are there any $i \neq j \in [n] \times [n]$ such that $f(i) = f(j)$?
- Query complexity: $\mathcal{O}(n^{2/3})$ (Ambainis 2003) $\Omega(n^{2/3})$ (Aaronson & Shi 2001)

Triangle Finding **[(2014) non-walk algorithm by Le Gall: $\widetilde{\mathcal{O}}(n^{5/4})$]**

- Black box: For any pair $u, v \in V \times V$ tells whether there is an edge uv
- Question: Is there any triangle in G?
- Query complexity: $\mathcal{O}(n^{13/10})$ (Magniez, Sántha, Szegedy 2003)

Walks on the Johnson graph (Sántha `arXiv:0808.0059`)

Vertices: $\{S \subset N : |S| = K\}$; Edges: $\{(S, S') : |S \triangle S'| = 2\}$

Element Distinctness

- Black box: Computes f on inputs corresponding to elements of $[n]$
- Question: Are there any $i \neq j \in [n] \times [n]$ such that $f(i) = f(j)$?
- Query complexity: $\mathcal{O}(n^{2/3})$ (Ambainis 2003) $\Omega(n^{2/3})$ (Aaronson & Shi 2001)

Triangle Finding [(2014) non-walk algorithm by Le Gall: $\widetilde{\mathcal{O}}(n^{5/4})$]

- Black box: For any pair $u, v \in V \times V$ tells whether there is an edge uv
- Question: Is there any triangle in G?
- Query complexity: $\mathcal{O}(n^{13/10})$ (Magniez, Sántha, Szegedy 2003)

Matrix Product Verification

- Black box: Tells any entry of the $n \times n$ matrices A, B or C.
- Question: Does $AB = C$ hold?
- Query complexity: $\mathcal{O}(n^{5/3})$ (Buhrman, Špalek 2004)

Continuous-time quantum walks

Continuous-time quantum / random walks

Laplacian of a weighted graph

Let $G = (V, E)$ be a finite simple graph, with non-negative edge-weights $w\colon E \to \mathbb{R}_+$. The Laplacian is defined as

$$u \neq v\colon\ L_{uv} = w_{uv},\ \text{ and } L_{uu} = -\sum_v w_{uv}.$$

Continuous-time quantum / random walks

Laplacian of a weighted graph

Let $G = (V, E)$ be a finite simple graph, with non-negative edge-weights $w\colon E \to \mathbb{R}_+$. The Laplacian is defined as

$$u \neq v\colon\ L_{uv} = w_{uv},\ \text{and}\ L_{uu} = -\sum_{v} w_{uv}.$$

Continuous-time walks

Evolution of the state:

$$\frac{d}{dt} p_u(t) = \sum_{v \in V} L_{uv} p_v(t) \qquad \Longrightarrow \qquad p(t) = e^{tL} p(0)$$

Continuous-time quantum / random walks

Laplacian of a weighted graph

Let $G = (V, E)$ be a finite simple graph, with non-negative edge-weights $w \colon E \to \mathbb{R}_+$. The Laplacian is defined as

$$u \neq v \colon L_{uv} = w_{uv}, \text{ and } L_{uu} = -\sum_v w_{uv}.$$

Continuous-time walks

Evolution of the state:

$$\frac{d}{dt} p_u(t) = \sum_{v \in V} L_{uv} p_v(t) \qquad \Longrightarrow \qquad p(t) = e^{tL} p(0)$$

$$i \frac{d}{dt} \psi_u(t) = \sum_{v \in V} L_{uv} \psi_v(t) \qquad \Longrightarrow \qquad \psi(t) = e^{-itL} \psi(0)$$

Exponential speedup by a quantum walk

Childs, Cleve, Deotto, Farhi, Gutmann, and Spielman: `quant-ph/0209131`

Same speed-up by Szegedy walks?

Can be reduced to the line

$$A = \frac{1}{3}\begin{pmatrix} 1 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 1 \end{pmatrix}$$

Show that the bottom left corner of $T_m(A)$ is $1/\mathrm{poly}(n)$ large for some $m = \mathrm{poly}(n)$.