Rich dynamics of qubits

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CTU CZECH TECHNICAL UNIVERSITY IN PRAGUE

TECHNISCHE UNIVERSITÄT DARMSTADT

CSRC

Quantum theory: linear or nonlinear?

1., Closed systems - unitary operators - linear evolution

2., Quantum channels - completely positive maps - linear evolution

If quantum states evolved nonlinearly

- then hard problems (NP complete)
  would be easily solved (in polynomial time)

  D. S. Abrams and S. Lloyd, Phys. Rev. Lett. 81, 3992 (1998)

- quick discrimination of nonorthogonal states - generic feature

  A. M. Childs and J. Young, Phys. Rev. A, 93, 022314 (2016)

- BEC + nonlinear quantum walks

  D. A. Meyer and T. G. Wong, New J. Phys. 15, 063014 (2013)

  A. Alberti and S. Wimberger, Phys. Rev. A 96, 023620 (2017)

  M. Maeda, H. Sasaki, E. Segawa, et al. Quantum. Inf. Process. 17, 215 (2018)

Effective nonlinearity in quantum theory

PHYSICAL REVIEW A **89**, 012312 (2014)

Quantum search with general nonlinearities

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(Received 13 November 2013; published 14 January 2014)

Evolution by the Gross-Pitaevskii equation, which describes Bose-Einstein condensates under certain conditions, solves the unstructured search problem more efficiently than does the Schrödinger equation, because it includes a cubic nonlinearity, proportional to $|\psi|^2\psi$. This is not the only nonlinearity of the form $f(|\psi|^2)\psi$ that arises in effective equations for the evolution of real quantum physical systems, however: The cubic-quintic nonlinear Schrödinger equation describes light propagation in nonlinear Kerr media with defocusing corrections, and the logarithmic nonlinear Schrödinger equation describes Bose liquids under certain conditions. Analysis of computation with such systems yields some surprising results; for example, when time-measurement precision is included in the resource accounting, searching a "database" when there is a single correct answer may be easier than searching when there are multiple correct answers. In each of these cases the nonlinear equation is an effective approximation to a multiparticle Schrödinger equation, for search by which Grover's algorithm is optimal. Thus our results lead to quantum information-theoretic bounds on the physical resources required for these effective nonlinear theories to hold, asymptotically.

Nonlinear transformations by selective evolution

3., Measurements

- ✓ projection (von Neumann)
- ✓ probabilistic (Born)
- ✓ information gained
- ☞ information feed-back, post-selection

↯ breaking linearity ↯

Quest for the most general quantum evolution

Linear maps

- Rules of standard, non-relativistic quantum mechanics
- Ancilla systems
- Many identical copies: ensemble

Quest for the most general quantum evolution

Ancilla from the same ensemble

- Operation on n identical systems
  — larger Hilbert space
- Measurement + post-selection on $n - 1$ systems
- Nonlinear evolution for the remaining system
  — same Hilbert space

Quest for the most general quantum evolution

Ancilla from the same ensemble

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The resulting nonlinear map can be

- Deterministic
- Pure states $\rightarrow$ pure states
- Complex, deterministic chaos may emerge

T. Kiss, I. Jex, G. Alber, S. Vymětal, Phys. Rev. A **74**, 040301(R) (2006).

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Possibilities to explore

1. Pure or mixed initial states
2. Closed or open evolution
3. $\dim \mathcal{H} = 2, 3, 4, \ldots, \infty$

Preliminaries

- General single-qubit pure state:

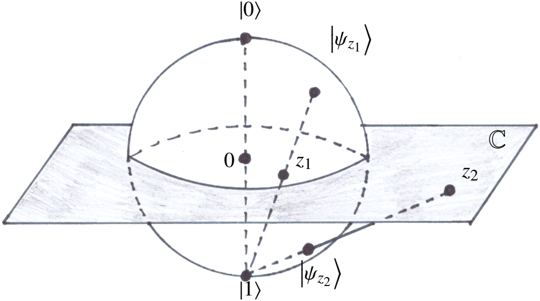
$$|\psi\rangle = N\,(a\,|0\rangle + b\,|1\rangle) \quad \text{where } a, b \in \mathbb{C}, \quad N = 1/\sqrt{|a|^2 + |b|^2}$$

- One can also write:

$$|\psi\rangle = N\,a\left(|0\rangle + \frac{b}{a}\,|1\rangle\right) = \frac{1}{\sqrt{1+|z|^2}}\,(|0\rangle + z\,|1\rangle) \quad \text{where } z \in \mathbb{C}_\infty = \mathbb{C} \cup \infty$$

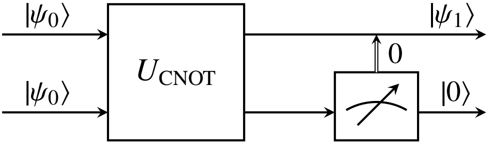
up to a global phase

- Visualization on complex plane ↔ Bloch sphere (Riemann sphere)



stereographic projection of the Bloch sphere onto the complex plane

Basic scheme for a nonlinear state transformation

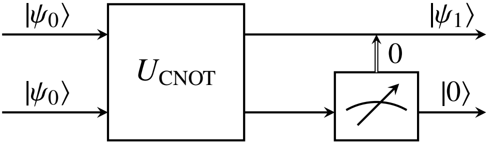


H. Bechmann-Pasquinucci, B. Huttner, & N. Gisin Phys. Lett. A **242**, 198 (1998).

Basic scheme for a nonlinear state transformation

$$|\psi_0\rangle = \frac{|0\rangle + z\,|1\rangle}{\sqrt{1+|z|^2}}$$

$(z \in \mathbb{C}_\infty)$



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$$\left|\Psi^{\mathrm{in}}\right\rangle = |\psi_0\rangle \otimes |\psi_0\rangle = \frac{1}{1+|z|^2}\left(|00\rangle + z\,|01\rangle + z\,|10\rangle + z^2\,|11\rangle\right)$$

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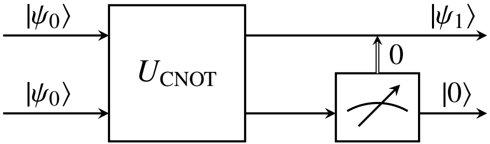

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$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\left|\Psi^{\text{out}}\right\rangle = U_{\text{CNOT}}\left|\Psi^{\text{in}}\right\rangle = \frac{1}{1+|z|^2}\left(|00\rangle + z\,|01\rangle + z\,|11\rangle + z^2\,|10\rangle\right)$$

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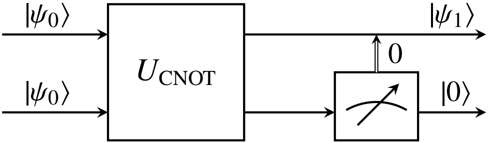

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- after projecting the target qubit to $|0\rangle$:

$$|\psi_1\rangle = \frac{1}{\sqrt{1+|z|^4}}\Big(|0\rangle + z^2|1\rangle\Big)$$

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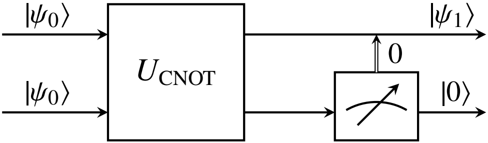

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- after projecting the target qubit to $|0\rangle$:

$$|\psi_1\rangle = \frac{1}{\sqrt{1+|z|^4}}\Big(|0\rangle + z^2\,|1\rangle\Big) \longrightarrow f(z) = z^2$$

Iterations of the scheme

- We assume many qubits in the same initial state $|\psi_0\rangle$ (ensemble)
- We form pairs of them, and apply the scheme $\Rightarrow$ iterate the map
- For higher iterates we need all previous steps to succeed

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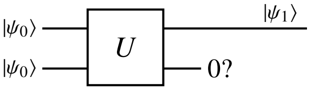

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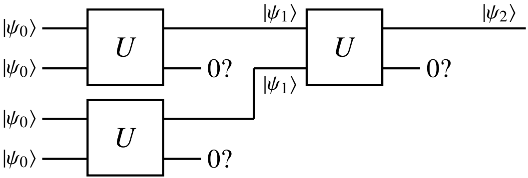

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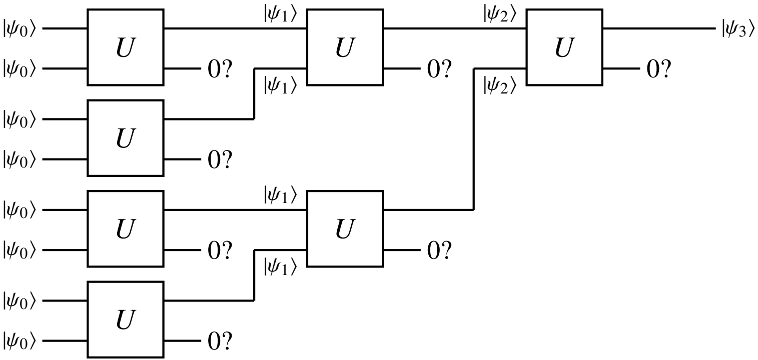
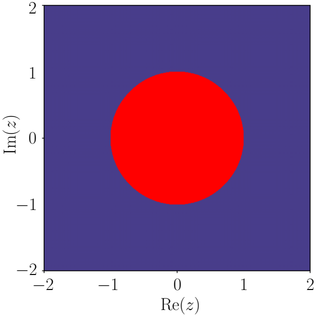


Iterations of the nonlinear map

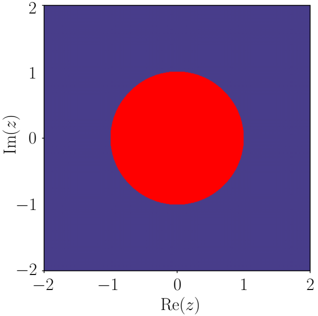


Iteration of $f(z) = z^2$ on the complex plane:

- $|z| < 1 \to 0$ (fixed point)
- $|z| > 1 \to \infty$ (fixed point)
- Julia set: $|z| = 1$ unit circle
  - chaotic dynamics
  - contains repelling cycles

J. Milnor, *Dynamics in One Complex Variable* (Princeton University Press, 2006)

Iterations of the nonlinear map



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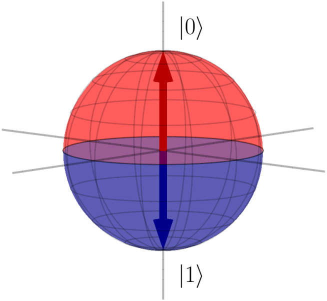

Iteration of $f(z) = z^2$ on the Bloch sphere:

- if $|z| < 1$ states converge to $|0\rangle$
- if $|z| > 1$ states converge to $|1\rangle$
- Julia set: equator
  - equally separates regions of convergence

Iterative nonlinear quantum protocols

- ensemble of qubits - quantum gates - measurement
- all basic components of a universal quantum circuit
- iterate the protocol

Complex chaos

- deterministic
- pure quantum states remain pure
- positive Lyapunov exponent

T. Kiss, I. Jex, G. Alber, S. Vymětal, Phys. Rev. A **74**, 040301(R) (2006).

An application for quantum state discrimination

PHYSICAL REVIEW A **95**, 023828 (2017)

Measurement-induced chaos and quantum state discrimination in an iterated Tavis-Cummings scheme

Juan Mauricio Torres,[1,2] József Zsolt Bernád,[1] Gernot Alber,[1] Orsolya Kálmán,[3] and Tamás Kiss[3]

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[2] *Instituto de Física, Benemérita Universidad Autónoma de Puebla, Apdo. Postal J-48, Puebla, Pue. 72570, México*

[3] *Institute for Solid State Physics and Optics, Wigner Research Centre, Hungarian Academy of Sciences, P.O. Box 49, H-1525 Budapest, Hungary*

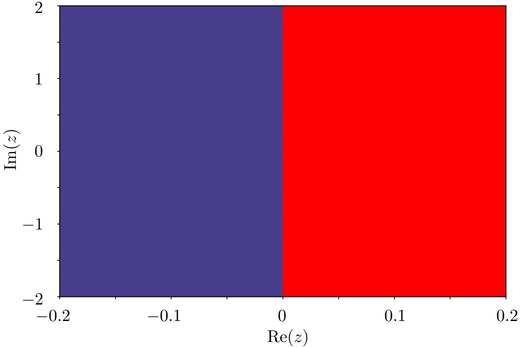
(Received 7 October 2016; published 14 February 2017)

A cavity quantum electrodynamical scenario is proposed for implementing a Schrödinger microscope capable of amplifying differences between nonorthogonal atomic quantum states. The scheme involves an ensemble of identically prepared two-level atoms interacting pairwise with a single mode of the radiation field as described by the Tavis-Cummings model. By repeated measurements of the cavity field and of one atom within each pair a measurement-induced nonlinear quantum transformation of the relevant atomic states can be realized. The intricate dynamical properties of this nonlinear quantum transformation, which exhibits measurement-induced chaos, allow approximate orthogonalization of atomic states by purification after a few iterations of the protocol and thus the application of the scheme for quantum state discrimination.

DOI: 10.1103/PhysRevA.95.023828

An application for quantum state discrimination

- if instead of CNOT, we apply $U = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$
- nonlinear transformation: $f = \dfrac{2z}{1+z^2}$



- two superattractive fixed points: 1 and -1
- Julia set: imaginary axis

J. M. Torres, J. Z. Bernád, G. Alber, O. Kálmán, and T. Kiss, Phys. Rev. A, **95**, 023828 (2017)

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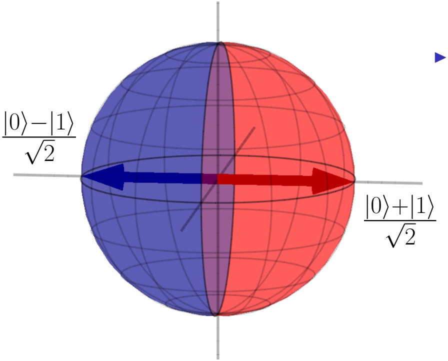

- Julia set: longitudinal great circle through y axis
  - equally separates regions of convergence

J. M. Torres, J. Z. Bernád, G. Alber, O. Kálmán, and T. Kiss, Phys. Rev. A, **95**, 023828 (2017)

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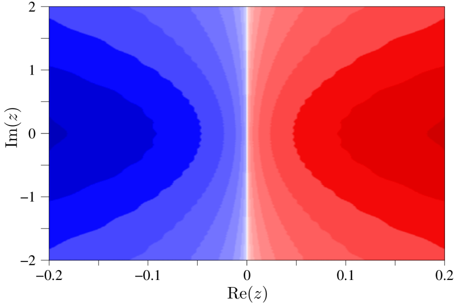
- nonlinear transformation: $f = \dfrac{2z}{1+z^2}$
- From highly overlapping to almost orthogonal in only 3 steps



$$|\Psi_0\rangle_I = \frac{|0\rangle + 0.2\,|1\rangle}{\sqrt{1+(0.2)^2}}$$

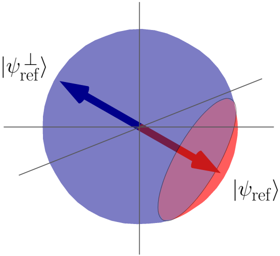
$$|\Psi_0\rangle_{II} = \frac{|0\rangle - 0.2\,|1\rangle}{\sqrt{1+(0.2)^2}}$$

$${}_I\langle\Psi_0\,|\,\Psi_0\rangle_{II} \approx 0.92$$

$${}_I\langle\Psi_3\,|\,\Psi_3\rangle_{II} \approx 0.08$$

G. Zhu, O. Kálmán, K. Wang, L. Xiao, D. Qu, X. Zhan, Z. Bian, T. Kiss, P. Xue, PRA, **100**, 052307 (2019)

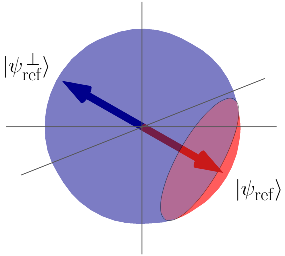
Quantum state matching



- define a reference state: $|\psi_{\mathrm{ref}}\rangle$
- define a neighborhood: $\varepsilon = |\langle\psi \mid \psi_{\mathrm{ref}}\rangle|$
- find which f corresponds to it
- find implementation of f
  - 2-qubit unitary+post-selection

O. Kálmán and T. Kiss, Phys. Rev. A **97**, 032125 (2018)

Quantum state matching



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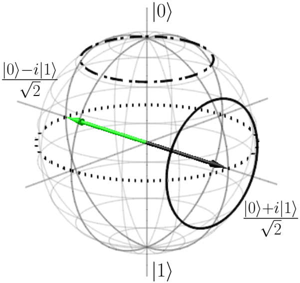
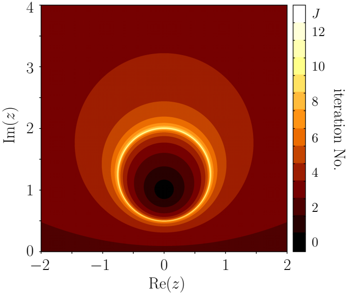




$$|\psi_{\rm ref}\rangle = \frac{|0\rangle + i\,|1\rangle}{\sqrt{2}}$$

$$\varepsilon^2 = 0.9$$

O. Kálmán and T. Kiss, Phys. Rev. A **97**, 032125 (2018)

Iterative nonlinear quantum protocols

- Ensemble of qubits in *pure state* $|\psi_0\rangle \sim |0\rangle + z\,|1\rangle \quad (z \in \mathbb{C})$
  1. Take them pairwise: $|\Psi_0\rangle = |\psi_0\rangle_A \otimes |\psi_0\rangle_B$
  2. Apply an **entangling** two-qubit operation U
  3. Measure the state of qubit B — keep A only for result 0
- Smaller ensemble in *pure state* $|\psi_1\rangle \sim |0\rangle + f(z)\,|1\rangle$
- Quantum magnification bound: exponential downscaling of the ensemble

$$U \leftrightarrow f(z) = \frac{a_0 z^2 + a_1 z + a_2}{b_0 z^2 + b_1 z + b_2}$$

A. Gilyén, T. Kiss and I. Jex, Sci. Rep. **6**, 20076 (2016)

Historical remarks on complex dynamics

- ☞ Iterated rational polynomials: $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}, \quad f^{\circ n} \to ?$
- ☞ One century of complex chaos:

1871 idea of iterated functions by Ernst Schröder

Ueber iterirte Functionen., Math. Ann.

1906 first weird example by P. Fatou: $z \mapsto z^2/(z^2+2)$

1920ies G. Julia, S. Lattès, & . . .

1970ies Computers help visualize: B. Mandelbrot & . . .

- ☞ A good book:
J.W. Milnor *Dynamics in One Complex Variable*, (Vieweg, 2000)

Iterative dynamics - examples

CNOT gate plus a single qubit gate

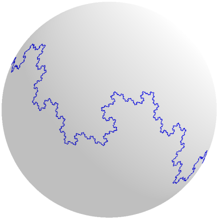
$$U = \begin{pmatrix} \cos\theta & \sin\theta\, e^{i\varphi} \\ -\sin\theta\, e^{-i\varphi} & \cos\theta \end{pmatrix}$$

Family of maps over $\mathbb{C}_\infty$:

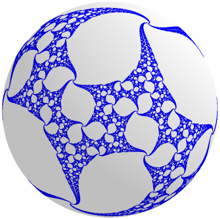
$$z \mapsto f_p(z) = \frac{z^2 + p}{1 - p^* z^2} \qquad p = \tan\theta\, e^{i\varphi}$$

$p \in \mathbb{C}$ parameter of the gate

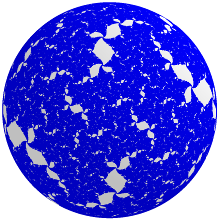
Iterative dynamics - Julia sets on the Bloch sphere



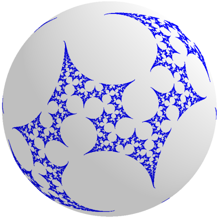
(a) $\theta = 0.4, \varphi = \frac{\pi}{2}$



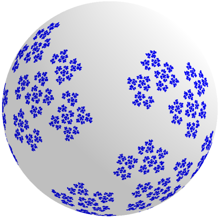
(b) $\theta = 0.55, \varphi = \frac{\pi}{2}$



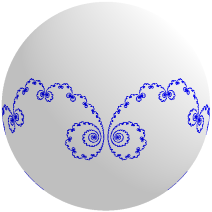
(c) $\theta = 0.633, \varphi = \frac{\pi}{2}$



(d) $\theta = 1.05, \varphi = \frac{\pi}{2}$



(e) $\theta = 0.5, \varphi = 0.5$

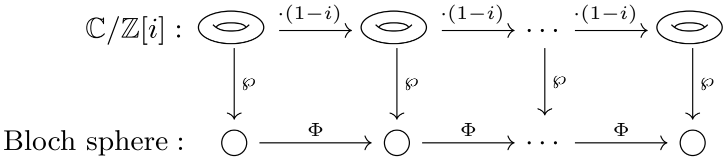


(f) $\theta = 0.232, \varphi = 0$

A. Gilyén, T. Kiss and I. Jex, Sci. Rep. **6**, 20076 (2016).

Lattès map: $J = \hat{\mathbb{C}}$

$$f(z) = \frac{z^2 + i}{iz^2 + 1}, \quad p = i$$



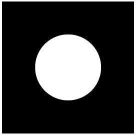
A commutative diagram:

- map on the Bloch sphere $\leftrightarrow \times(1-i)^n$ on the torus
- all initial states are weird
- ergodicity

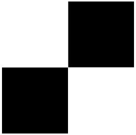
Lattès, S (1918), Les Comptes rendus de l'Académie des sciences, 166: 26-28

A. Gilyén, T. Kiss and I. Jex, Sci. Rep. **6**, 20076 (2016).

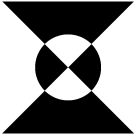
Lattès map: ergodic dynamics



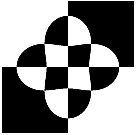
(a) $|z| > 1$



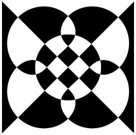
(b) $|f(z)| > 1$



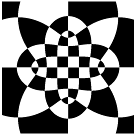
(c) $|f^{\circ 2}(z)| > 1$



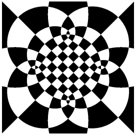
(d) $|f^{\circ 3}(z)| > 1$



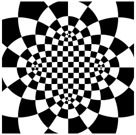
(e) $|f^{\circ 4}(z)| > 1$



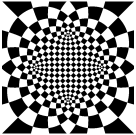
(f) $|f^{\circ 5}(z)| > 1$



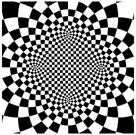
(g) $|f^{\circ 6}(z)| > 1$



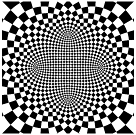
(h) $|f^{\circ 7}(z)| > 1$



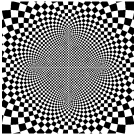
(i) $|f^{\circ 8}(z)| > 1$



(j) $|f^{\circ 9}(z)| > 1$

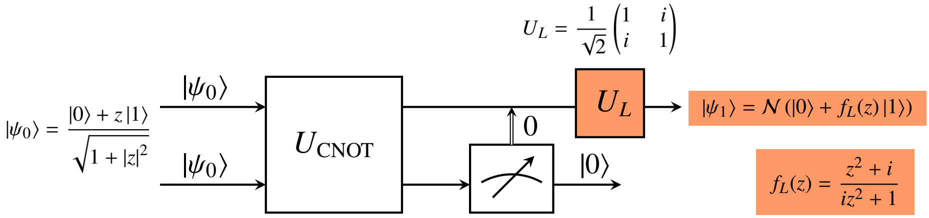


(k) $|f^{\circ 10}(z)| > 1$

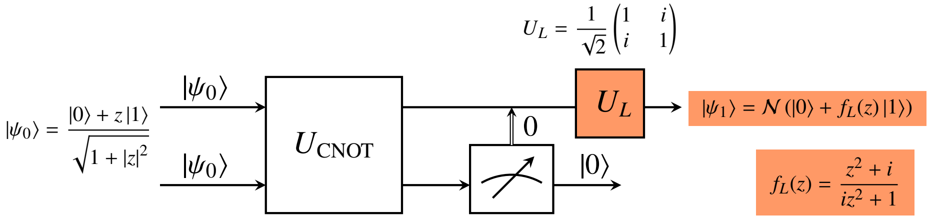


(l) $|f^{\circ 11}(z)| > 1$

Scheme resulting in an ergodic (Lattès) map



Scheme resulting in an ergodic (Lattès) map

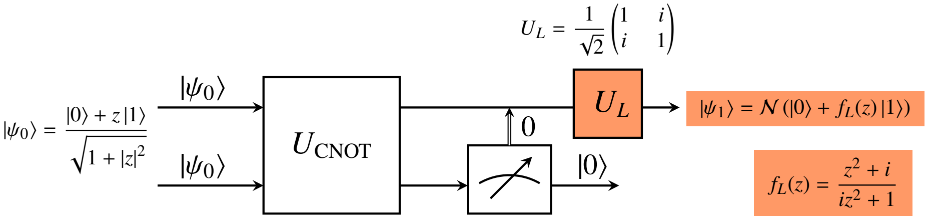


Iterative dynamics of pure initial states

- no attractive fixed cycles
- all pure initial states $\in$ Julia set
- every initial state is chaotic

A. Gilyén, T. Kiss and I. Jex, Sci. Rep. **6**, 20076 (2016).

Scheme resulting in an ergodic (Lattès) map



Iterative dynamics of pure initial states

- no attractive fixed cycles
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A. Gilyén, T. Kiss and I. Jex, Sci. Rep. **6**, 20076 (2016).

Does noise destroy this property?

Lattès map with noisy initial states

Dynamics represented by $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ functions:

$$u' = \frac{u^2 - v^2}{1 + w^2}, \quad v' = \frac{2w}{1 + w^2}, \quad w' = -\frac{2uv}{1 + w^2}$$

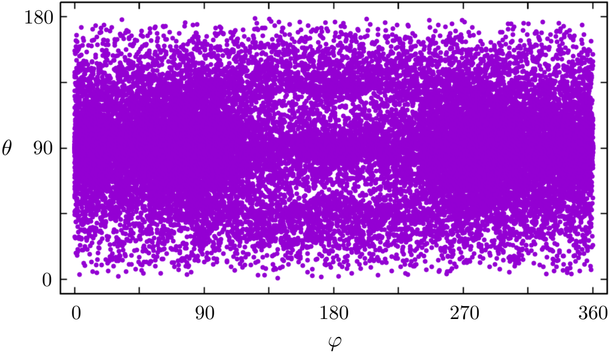
No book by Milnor! :-(

Asymptotics: all mixed initial states → completely mixed state

O. Kálmán, T. Kiss and I. Jex, J Russ Laser Res 39: 382 (2018)

Is there an ergodic regime for noisy initial states?

Evolution of slightly perturbed initial states

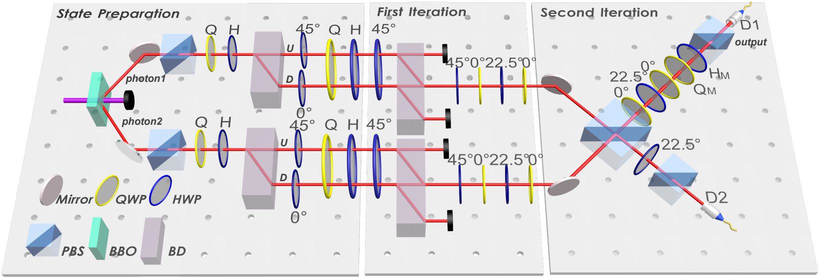


9 iterative steps

$r_0 = 0.99,\ \Delta(\theta_0, \varphi_0) \sim 10^{-3},\ r_{min} \sim 0.5$

D. Qu, O. Kálmán, G. Zhu, L. Xiao, K. Wang, T. Kiss, and P. Xue, New J Phys. **23** 083008 (2021)

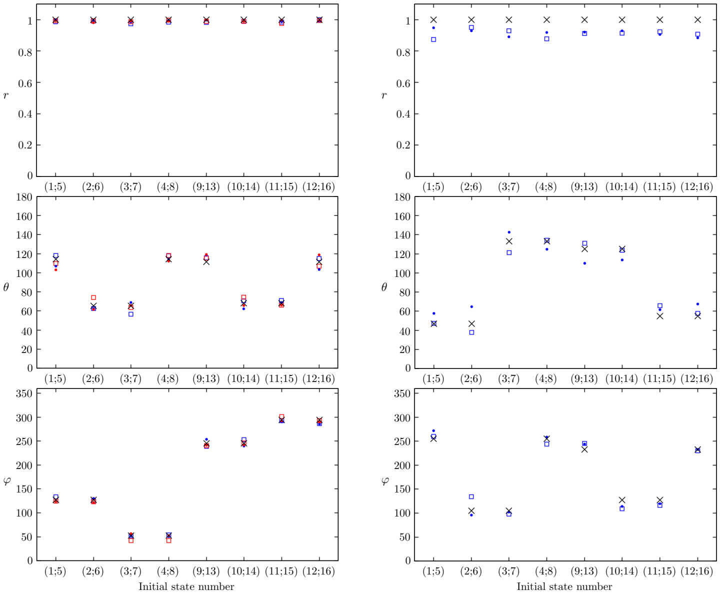
Optical realization of the ergodic protocol



- 2 true steps of the scheme were realized
- down-converted photons from BBO crystal
- initial qubit states are encoded into:
  - the polarization degree of freedom and
  - the path degree of freedom of photons

D. Qu, O. Kálmán, G. Zhu, L. Xiao, K. Wang, T. Kiss, and P. Xue, New J Phys. **23** 083008 (2021)

Optical realization of the ergodic protocol

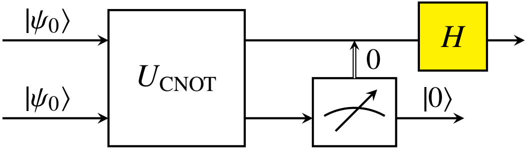


D. Qu, O. Kálmán, G. Zhu, L. Xiao, K. Wang, T. Kiss, and P. Xue, New J Phys. **23** 083008 (2021)

Scheme with CNOT & Hadamard gate

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

$$|\psi_0\rangle = \frac{|0\rangle + z\,|1\rangle}{\sqrt{1+|z|^2}}$$



$$|\psi_1\rangle = \mathcal{N}\left(|0\rangle + f_H(z)\,|1\rangle\right)$$

$$f_H(z) = \frac{1-z^2}{1+z^2}$$

Scheme with CNOT & Hadamard gate

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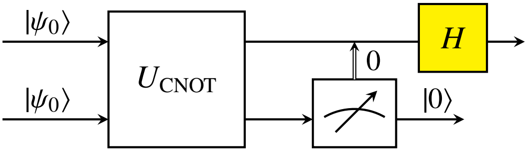

$$|\psi_1\rangle = \mathcal{N}\left(|0\rangle + f_H(z)\,|1\rangle\right)$$

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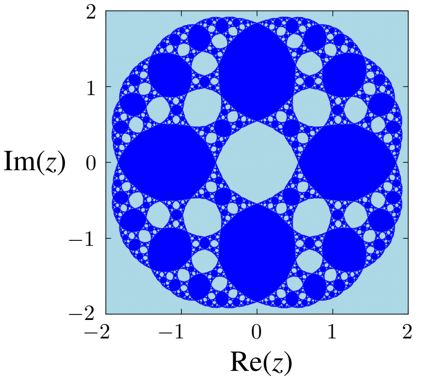
Iterative dynamics (pure initial states)



- attractive length-2 cycle:

$$\left|\psi^{(1)}\right\rangle = |0\rangle \quad (z=0)$$

$$\updownarrow$$

$$\left|\psi^{(2)}\right\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad (z=1)$$

- border: **Julia set**

CNOT + Hadamard gate: phase transition

Noisy (mixed) initial states:

$$\rho \xrightarrow{\mathcal{M}} \rho' = U_H \, \frac{\rho \odot \rho}{\mathrm{Tr}(\rho \odot \rho)} \, U_H^\dagger$$

where

$$U_H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \rho = \frac{1}{2} \begin{pmatrix} 1+w & u-iv \\ u+iv & 1-w \end{pmatrix}$$

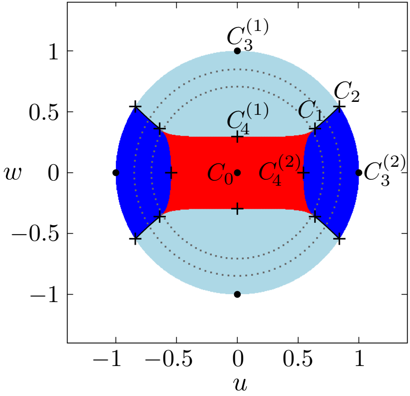
Purity: $P = \mathrm{Tr}(\rho^2) = (1 + u^2 + v^2 + w^2)/2 \leq 1$

Noisy initial states (Hadamard case)

$$\rho^{(0)} = \frac{1}{2}\begin{pmatrix} 1+w & u-iv \\ u+iv & 1-w \end{pmatrix} \xrightarrow{\mathcal{M}_H} \rho^{(1)} = \frac{1}{2}\begin{pmatrix} 1+w' & u'-iv' \\ u'+iv' & 1-w' \end{pmatrix}$$

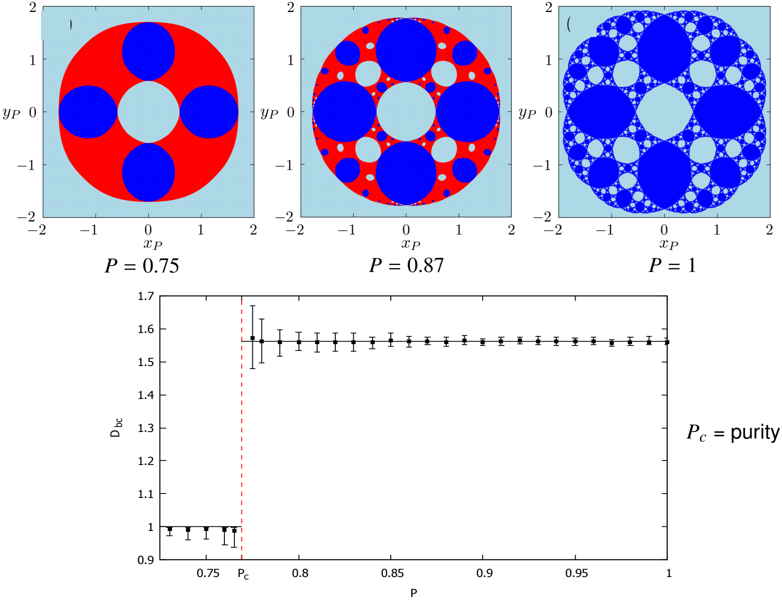
$$u' = \frac{2w}{1+w^2}, \quad v' = \frac{-2uv}{1+w^2}, \quad w' = \frac{u^2-v^2}{1+w^2}$$

- $(u, 0, w)$ invariant plane inside the Bloch sphere



Fixed cycles of $\mathcal{M}_H$	Purity
C_0	0.5
C_1	0.769
C_2	1
$C_3^{(1)} \leftrightarrow C_3^{(2)}$	$1 \leftrightarrow 1$
$C_4^{(1)} \leftrightarrow C_4^{(2)}$	$0.648 \leftrightarrow 0.544$

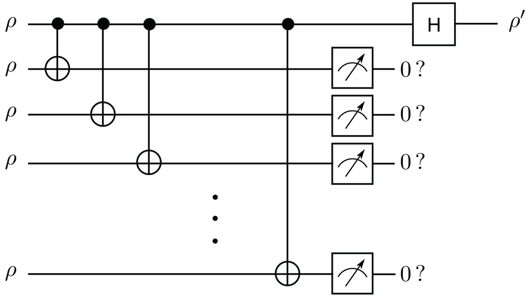
Fractal dimension D_{bc} as a function of purity P



P_c = purity of C_1

M. Malachov, I. Jex, O. Kálmán, T. Kiss, Chaos **29**, 033107 (2019).

Nonlinear (Hadamard) protocols of order $n > 2$



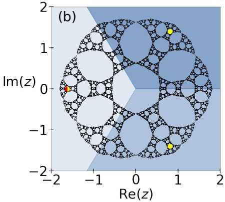
- Map of the pure case: $f_n(z) = \dfrac{1 - z^n}{1 + z^n}$
- Map for noisy states:

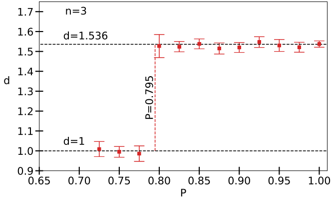
$$u' = \frac{(1+w)^n - (1-w)^n}{(1+w)^n + (1-w)^n}, \quad v' = -\frac{2\mathrm{Im}\left[(u+iv)^n\right]}{(1+w)^n + (1-w)^n}, \quad w' = \frac{2\mathrm{Re}\left[(u+iv)^n\right]}{(1+w)^n + (1-w)^n}$$

A. Portik, O. Kálmán, I. Jex, T. Kiss, Phys. Lett. A **431**, 127999 (2022).

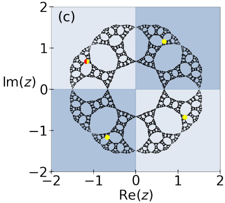
Nonlinear (Hadamard) protocols of order $n > 2$

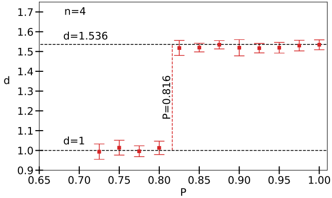
$n = 3$



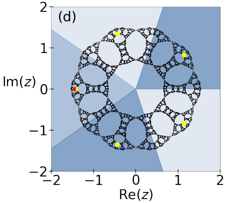


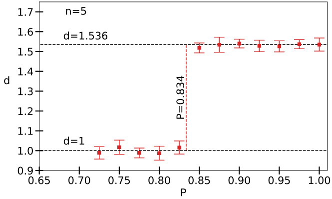
$n = 4$





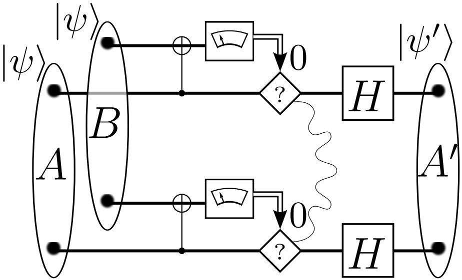
$n = 5$





A. Portik, O. Kálmán, I. Jex, T. Kiss, Phys. Lett. A **431**, 127999 (2022).

LOCC scheme with 2 qubits



$$|\psi\rangle = c_1|00\rangle + c_2|01\rangle + c_3|10\rangle + c_4|11\rangle$$

$$|\psi'\rangle = U_H \otimes U_H \left[\mathcal{N}(c_1^2|00\rangle + c_2^2|01\rangle + c_3^2|10\rangle + c_4^2|11\rangle)\right]$$

2 qubits: chaotic entanglement

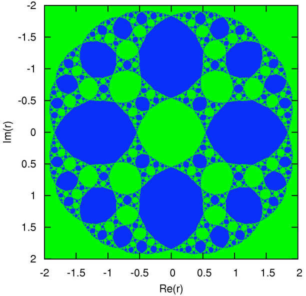
Asymptotic states

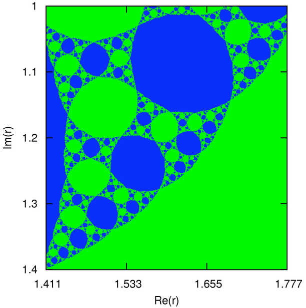
Green: Fully entangled:

$$|\psi^{(\infty)}\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle\right)$$

Blue: Completely separable, oscillatory:

$$|\psi^{(\infty)}\rangle \to \left\{ |00\rangle \, , \, \frac{1}{2} \left(|0\rangle + |1\rangle\right) \otimes \left(|0\rangle + |1\rangle\right) \right\}$$





T. Kiss, S. Vymětal, L. D. Tóth, A. Gábris,
I. Jex, G. Alber, PRL **107**, 100501 (2011)

Outlook

- Nonlinear protocols may be used for benchmarking QCs
  - First implementation of a nonlinear protocol (Lattès) for benchmarking by **András Gilyén** et al.

    A. Cornelissen, J. Bausch, and A. Gilyén, arXiv:2104.10698 (2021)
  - Matching an unknown qubit state to a reference qubit state
    - original scheme: O. Kálmán and T. Kiss, Phys. Rev. A **97**, 032125 (2018)
- Concentration & distillation of Bell pairs
- Networks: e. g. GHZ states

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